# Invention Disclosure

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**Preamble:**

Bancor is a liquidity protocol that accepts cryptocurrency from users and utilizes defined algorithms to participate in financial markets on their behalf. The present invention is a profound departure from Bancor's existing product, including its recent Version 3 release. The invention includes a description of a novel market making algorithm that underpins the exchange of tokens between different participants within a distributed application environment, such as a blockchain, in addition to a description of its smart contract implementation.

**Context:**

Decentralized finance (DeFi) is a distributed system of applications built on public blockchains that fulfill the need for economic and financial primitives for cryptocurrency. Decentralized exchanges (DEXes) are protocols that allow for cryptocurrency tokens to be exchanged between users in a permissionless manner. The prototypical example is a smart contract (“liquidity pool”) that collects crowd-sourced capital from users (“liquidity providers”) and issues a receipt token (“pool token”) in return. The pool token represents the liquidity provider’s pro-rata share of the liquidity pool as a whole; therefore, any proportion of the pool token supply is redeemable for the same proportion of the liquidity pool. The composition of the liquidity pool may change over time as users (“traders”) exchange their own tokens for those inside the liquidity pool. An important mainstay of DeFi is the invariant function DEX, where exchange rates are determined by equations which force the composition of the liquidity pool to adhere to a predefined profile (“bonding curve”). Preeminent examples are constant product, and constant sum; however, more elaborate variations of these concepts have appeared which have fine-tuned the behavior of their protocols and cognate liquidity pools. The “value” function, and the “stableswap” function popularized by Balancer Protocol, and Curve Finance, respectively, are noteworthy modifications of common archetypes.

Importantly, liquidity providers on DEXes are beholden to parameters of the liquidity pool to which they contribute their tokens. Thus, the agency of an individual liquidity provider to decide their own exchange rates and execute precise trading strategies is suppressed in favor of a prescribed, generic interpretation of their intentions. In the broadest sense, the present invention offers a novel means of liquidity provision whereby users can exercise their personal trading preferences. This is made possible by a customizable invariant function that encompasses an infinite set of bonding curves, the elements of which include both constant sum and constant product models. Each liquidity provider therefore owns their own liquidity pool, for which they may be the only participant. Moreover, each liquidity pool is governed by a set of bonding curves, instead of just one, which allows for the tokens to be deployed asymmetrically. In contrast to the industry status quo where tokens sold by a liquidity pool are assumed to be purchased back by the pool at the same exchange rate, the asymmetry paradigm of the present invention allows for the sale and repurchase rates, as well as their relative acceleration, to be independent of each other.

In the common vernacular, the present invention caters to a “buy low, sell high” trading mentality, and allows its liquidity providers to determine their own unique pool characteristics, supported by an infinitely customizable bonding curve and associated smart contract infrastructure. Each pool and its contents can variably be the sole property of its creator, or, alternatively, be open for community participation through the issuance of its own pool tokens. The latter describes a novel type of subscriber trading product and represents a decentralized counterpart to copy trading on conventional, centralized exchanges.

Taken together, the improvements offered by the invention described herein are both novel and address recognizable industry pain points that have greatly limited the scope of application for DEX protocols.

**Summary of the Invention:**

To achieve the features of the new product offering, we introduce 1) a novel invariant function and infinite set of associated bonding curves and 2) a novel asymmetric liquidity pool design employing one or more instances of these bonding curves supporting arbitrary trading strategies across any number of cryptocurrency tokens within the same pool. The following section provides a high-level summary of the key innovations that comprise a new technology.

1) **Novel Invariant Function.** A novel invariant function is employed with adjustable parameters that allow for users to determine the behavior of its associated liquidity pool with respect to exchange rates and price acceleration for all liquidity contained therein. In contrast to prior art, the function is infinitely customizable; it is limited in scope, accuracy, and resolution only by the virtual machine of the distributed ledger system it is deployed on.
2) **Asymmetric Liquidity Pools.** A novel liquidity pool design is employed which can use a plethora of bonding curves simultaneously to achieve an explicit trading profile, concordant with the objectives and bias of its creator. In contrast to prior art, asymmetric liquidity pools allow for decisive actions by liquidity providers. The newfound utility offers the means to execute premeditated plans, while maintaining liquidity for the cryptocurrency token economy.

**Background**

The *x* and *y* terms are intended as representations of balances of two assets with which an exchange will, or can, be performed. The *x* and *y* balances may be hypothetical, virtual, or real, and may represent any fungible, tokenized form of value on a distributed ledger, including but not necessarily limited to blockchains and directed acyclic graphs. The novel function described in the following equations, and which comprises a critical component of the overall invention, is necessarily defined with reference to the constant product function - a hyperbola with asymptotes at $x = 0$ and $y = 0$, and which is a subset of exchange algorithms featured in our prior art (**Appendix 1**).

The implicit curve of the constant product function can be written as $xy = k$ (**eqn. 1a**), or via its rearrangements (**eqns. 1b-c**). In the context of token balances in an associated smart contract, where this function is employed to perform exchange, its first derivative (**eqns. 1d-e**) can be used to evaluate the instantaneous rate, or exchange value, between any two tokens under its influence. A secant line connecting two points of the implicit curve, separated by a continuum of intermediate points, determines the absolute change in the smart contract holdings of the tokens represented by *x* and *y* (**eqns. 1f-g**). Therefore, the realized rate of exchange between any two tokens by a user of the system can be derived (**eqns. 1h-i**).

$$xy = k$$ **eqn. 1a**

$$y = \frac{k}{x}$$ **eqn. 1b**

$$x = \frac{k}{y}$$ **eqn. 1c**

$$-\frac{dy}{dx} = \frac{k}{x^2}$$ **eqn. 1d**

$$-\frac{dy}{dx} = \frac{y^2}{k}$$ **eqn. 1e**

$$-\Delta y = \frac{\Delta x k}{x\left(\Delta x + x\right)}$$ **eqn. 1f**

$$-\Delta y = \frac{\Delta x y^2}{\Delta x y + k}$$ **eqn. 1g**

$$-\frac{\Delta y}{\Delta x} = \frac{y^2}{\Delta x y + k}$$ **eqn. 1h**

$$-\frac{\Delta y}{\Delta x} = \frac{k}{x\left(\Delta x + x\right)}$$ **eqn. 1i**

It is from these concepts that the core utility of the novel algorithm, and its associated AMM design, are derived. In the section immediately following this text, where the properties of the novel invariant function are disclosed in detail, it may be appropriate to let each term be handled entirely in abstraction. The specific meaning with respect to the novel AMM design is the subject of detailed discussion in its own section.

**Description of the Invention:** *Novel Invariant Function*

What follows is a full mathematical elaboration of the novel invariant function, its bonding curves, and properties. The list of terms used to describe the function appears in **Table 1**; the equations wherein these terms are used are presented thereafter. The equations define each of these terms in the context of the novel function and its associated curves set; the graphical representations of these same mathematical concepts are depicted in example figures thereafter.

**Table 1.** Glossary of terms used to describe and characterize the novel invariant function disclosed herein.

| **Variable** | **Meaning** |
|---|---|
| $x$ | Pool state variable; virtual risk asset liquidity |
| $y$ | Pool state variable; virtual numeraire liquidity |
| $n$ | Curve parameter |
| $u$ | Curve parameter |
| $k$ | Curve parameter |
| $B$ | Curve parameter; square root of the gradient at $x = x_{\text{int}}$, $y = 0$ |
| $P$ | Curve parameter; gradient at $x = x_0$, $y = y_0$ (geometric mean price) |
| $Q$ | Curve parameter |
| $R$ | Curve parameter |
| $S$ | Curve parameter; difference of square roots of the gradients at the intercepts. |
| $x_0$ | Curve parameter; $x$-coordinate (risk asset) at geometric mean price |
| $y_0$ | Curve parameter; $y$-coordinate (numeraire) at geometric mean price |
| $x_{asym}$ | Curve parameter; $x$-asymptote, or vertical asymptote (risk asset) |
| $y_{asym}$ | Curve parameter; $y$-asymptote, or horizontal asymptote (numeraire) |
| $P_a$ | Curve parameter; gradient at $x = 0$, $y = y_{\text{int}}$ (price at high bound) |
| $P_b$ | Curve parameter; gradient at $x = x_{\text{int}}$, $y = 0$ (price at low bound) |
| $x_{int}$ | Curve parameter; $x$-coordinate at $y = 0$ ($x$-intercept of the risk asset) |
| $y_{int}$ | Curve parameter; $y$-coordinate at $x = 0$ ($y$-intercept of the numeraire) |

**The function underpinning this invention can be defined as:**

1) **A function with arbitrary, or adjustable asymptotes, and**
2) **Where the graph of the function shares at least one common point with eqns. 1a-c., and**
3) **Where the slope of the curve at the common point is identical with eqns. 1d-e.**

This definition is now elaborated. The relationship between *x* and *y* is under the influence of the novel invariant function, which can be written in many equivalent forms. The content provided herein in submitted as a representative sample. Therefore, the explicit forms that appear below include, but do not limit the scope of the disclosed invention. A representative set of novel invariant function appears in **eqns. 2a-p**, and examples of its algebraic manipulation into equivalent forms appear later in this document. In addition to the two variables, *x* and *y*, the function is adequately defined with respect to a minimum of three parameters from the set $n$, $u$, $k$, $P$, $Q$, $x_0$, $y_0$, $x_{\mathrm{asym}}$, $y_{\mathrm{asym}}$, $x_{\mathrm{int}}$, and $y_{\mathrm{int}}$. The function can also be defined with respect to the parameters $P_a$ and $P_b$, in addition to other rearrangements of these parameters, which is subject to further discussion elsewhere in this document.

$$k = \frac{x_0 y \left(nx + x_0 \left(1-n\right)\right)}{x_0\left(2-n\right) - x\left(1-n\right)} = \frac{x y_0 \left(ny + y_0\left(1-n\right)\right)}{y_0\left(2-n\right) - y\left(1-n\right)} \qquad \textbf{eqn. 2a}$$

$$P = \frac{y\left(nx + x_0\left(1-n\right)\right)}{x_0\left(x_0\left(2-n\right) - x\left(1-n\right)\right)} = \frac{y_0\left(y_0\left(2-n\right) - y\left(1-n\right)\right)}{x\left(ny + y_0\left(1-n\right)\right)} \qquad \textbf{eqn. 2b}$$

$$Q = \frac{\left(xy - x_0 y_0\right)}{\left(x - x_0\right)\left(y - y_0\right)} = \frac{x_{asym} y_{asym}}{\left(x - x_{asym}\right)\left(y - y_{asym}\right)} = \frac{xy}{\left(x - x_{int}\right)\left(y - y_{int}\right)} \qquad \textbf{eqn. 2c}$$

$$x_0\left(y_0\left(2-n\right) - y\left(1-n\right)\right) = x\left(ny + y_0\left(1-n\right)\right) \qquad \textbf{eqn. 2d}$$

$$y_0\left(x_0\left(2-n\right) - x\left(1-n\right)\right) = y\left(nx + x_0\left(1-n\right)\right) \qquad \textbf{eqn. 2e}$$

$$x_0 y_0 = \left(nx + x_0\left(1-n\right)\right)\left(ny + y_0\left(1-n\right)\right) \qquad \textbf{eqn. 2f}$$

$$xy = \left(x_0\left(2-n\right) - x\left(1-n\right)\right)\left(y_0\left(2-n\right) - y\left(1-n\right)\right) \qquad \textbf{eqn. 2g}$$

$$xy = \left(1-n\right)\left(x - x_0\right)\left(y - y_0\right) + x_0 y_0 \qquad \textbf{eqn. 2h}$$

$$x_{asym} y_{asym} = \left(1-n\right)^2\left(x - x_{asym}\right)\left(y - y_{asym}\right) \qquad \textbf{eqn. 2i}$$

$$xy = \left(1-n\right)^2\left(x - x_{int}\right)\left(y - y_{int}\right) \qquad \textbf{eqn. 2j}$$

$$x_0\left(y_0\left(\sqrt{Q}+1\right) - y\sqrt{Q}\right) = x\left(y\left(1-\sqrt{Q}\right) + y_0\sqrt{Q}\right) \qquad \textbf{eqn. 2k}$$

$$y_0\left(x_0\left(\sqrt{Q}+1\right) - x\sqrt{Q}\right) = y\left(x\left(1-\sqrt{Q}\right) + x_0\sqrt{Q}\right) \qquad \textbf{eqn. 2l}$$

$$x_0 y_0 = \left(x\left(1-\sqrt{Q}\right) + x_0\sqrt{Q}\right)\left(y\left(1-\sqrt{Q}\right) + y_0\sqrt{Q}\right) \qquad \textbf{eqn. 2m}$$

$$xy = \left(x_0\left(\sqrt{Q}+1\right) - x\sqrt{Q}\right)\left(y_0\left(\sqrt{Q}+1\right) - y\sqrt{Q}\right) \qquad \textbf{eqn. 2n}$$

$$xy = \sqrt{Q}\left(x - x_0\right)\left(y - y_0\right) + x_0 y_0 \qquad \textbf{eqn. 2o}$$

$$xy = Q\left(x - x_{int}\right)\left(y - y_{int}\right) \qquad \textbf{eqn. 2p}$$

Other forms of the novel invariant function include its various rearrangements, including and especially those which force either the *y*- or *x*- coordinates to be the subject (**eqns. 3a-n** and **4a-n**, respectively).

$$y = \frac{k\left(x_0\left(2-n\right)-x\left(1-n\right)\right)}{x_0\left(nx+x_0\left(1-n\right)\right)}$$ **eqn. 3a**

$$y = \frac{y_0\left(k\left(2-n\right)-xy_0\left(1-n\right)\right)}{k\left(1-n\right)+nxy_0}$$ **eqn. 3b**

$$y = \frac{Px_0\left(x_0\left(2-n\right)-x\left(1-n\right)\right)}{nx+x_0\left(1-n\right)}$$ **eqn. 3c**

$$y = \frac{y_0\left(y_0\left(2-n\right)-Px\left(1-n\right)\right)}{Pnx+y_0\left(1-n\right)}$$ **eqn. 3d**

$$y = \frac{y_0\left(x_0\left(2-n\right)-x\left(1-n\right)\right)}{nx+x_0\left(1-n\right)}$$ **eqn. 3e**

$$y = \frac{y_{asym}\left(x_{asym}+\left(1-n\right)^2\left(x-x_{asym}\right)\right)}{\left(1-n\right)^2\left(x-x_{asym}\right)}$$ **eqn. 3f**

$$y = \frac{y_{int}\left(1-n\right)^2\left(x_{int}-x\right)}{x-\left(1-n\right)^2\left(x-x_{int}\right)}$$ **eqn. 3g**

$$y = \frac{k\left(x_0\left(\sqrt{Q}+1\right)-x\sqrt{Q}\right)}{x_0\left(\left(1-\sqrt{Q}\right)x+x_0\sqrt{Q}\right)}$$ **eqn. 3h**

$$y = \frac{y_0\left(k\left(\sqrt{Q}+1\right)-xy_0\sqrt{Q}\right)}{k\sqrt{Q}+\left(1-\sqrt{Q}\right)xy_0}$$ **eqn. 3i**

$$y = \frac{Px_0\left(x_0\left(\sqrt{Q}+1\right)-x\sqrt{Q}\right)}{x\left(1-\sqrt{Q}\right)+x_0\sqrt{Q}}$$ **eqn. 3j**

$$y = \frac{y_0\left(y_0\left(\sqrt{Q}+1\right)-Px\sqrt{Q}\right)}{P\left(1-\sqrt{Q}\right)x+y_0\sqrt{Q}}$$ **eqn. 3k**

$$y = \frac{y_0\left(x_0\left(\sqrt{Q}+1\right)-x\sqrt{Q}\right)}{x\left(1-\sqrt{Q}\right)+x_0\sqrt{Q}}$$ **eqn. 3l**

$$y = \frac{y_{asym}\left(x_{asym}+Q\left(x-x_{asym}\right)\right)}{Q\left(x-x_{asym}\right)}$$ **eqn. 3m**

$$y = \frac{y_{int}Q\left(x_{int}-x\right)}{x-Q\left(x-x_{int}\right)}$$ **eqn. 3n**

$$x = \frac{x_0\left(k\left(2-n\right)-x_0y\left(1-n\right)\right)}{k\left(1-n\right)+nx_0y}$$ **eqn. 4a**

$$x = \frac{k\left(y_0\left(2-n\right)-y\left(1-n\right)\right)}{y_0\left(ny+y_0\left(1-n\right)\right)}$$ **eqn. 4b**

$$x = \frac{x_0\left(Px_0\left(2-n\right)-y\left(1-n\right)\right)}{P\left(x_0-nx_0\right)+ny}$$ **eqn. 4c**

$$x = \frac{y_0\left(y_0\left(2-n\right)-y\left(1-n\right)\right)}{P\left(ny+y_0\left(1-n\right)\right)}$$ **eqn. 4d**

$$x = \frac{x_0\left(y_0\left(2-n\right)-y\left(1-n\right)\right)}{ny+y_0\left(1-n\right)}$$ **eqn. 4e**

$$x = \frac{x_{asym}\left(y_{asym}+\left(n-1\right)^2\left(y-y_{asym}\right)\right)}{\left(n-1\right)^2\left(y-y_{asym}\right)}$$ **eqn. 4f**

$$x = \frac{x_{int}\left(1-n\right)^2\left(y_{int}-y\right)}{y-\left(1-n\right)^2\left(y-y_{int}\right)}$$ **eqn. 4g**

$$x = \frac{x_0\left(k\left(\sqrt{Q}+1\right)-x_0y\sqrt{Q}\right)}{k\sqrt{Q}+x_0y\left(1-\sqrt{Q}\right)}$$ **eqn. 4h**

$$x = \frac{k\left(y_0\left(\sqrt{Q}+1\right)-y\sqrt{Q}\right)}{y_0\left(y\left(1-\sqrt{Q}\right)+y_0\sqrt{Q}\right)}$$ **eqn. 4i**

$$x = \frac{x_0\left(Px_0\left(\sqrt{Q}+1\right)-y\sqrt{Q}\right)}{Px_0\left(1-\sqrt{Q}\right)\left(y-Px_0\right)}$$ **eqn. 4j**

$$x = \frac{y_0\left(y_0\left(\sqrt{Q}+1\right)-y\sqrt{Q}\right)}{P\left(y\left(1-\sqrt{Q}\right)+y_0\sqrt{Q}\right)}$$ **eqn. 4k**

$$x = \frac{x_0\left(y_0\left(\sqrt{Q}+1\right)-y\sqrt{Q}\right)}{y\left(1-\sqrt{Q}\right)+y_0\sqrt{Q}}$$ **eqn. 4l**

$$x = \frac{x_{asym}\left(y_{asym}+Q\left(y-y_{asym}\right)\right)}{Q\left(y-y_{asym}\right)}$$ **eqn. 4m**

$$x = \frac{x_{int}Q\left(y_{int}-y\right)}{y-Q\left(y-y_{int}\right)}$$ **eqn. 4n**

The novel invariant function can also be defined with respect to any line tangent to its implicit curve, measured from either the *x*- or *y*-coordinates (**eqns. 5a-n** and **6a-n**, respectively).

$$-\frac{dy}{dx} = \frac{k}{\left(nx + x_0\left(1-n\right)\right)^2}$$ **eqn. 5a**

$$-\frac{dy}{dx} = \frac{ky_0^2}{\left(k\left(1-n\right) + nxy_0\right)^2}$$ **eqn. 5b**

$$-\frac{dy}{dx} = \frac{Px_0^2}{\left(nx + x_0\left(1-n\right)\right)^2}$$ **eqn. 5c**

$$-\frac{dy}{dx} = \frac{Py_0^2}{\left(Pnx + y_0\left(1-n\right)\right)^2}$$ **eqn. 5d**

$$-\frac{dy}{dx} = \frac{x_0y_0}{\left(nx + x_0\left(1-n\right)\right)^2}$$ **eqn. 5e**

$$-\frac{dy}{dx} = \frac{x_{asym}y_{asym}}{\left(1-n\right)^2\left(x - x_{asym}\right)^2}$$ **eqn. 5f**

$$-\frac{dy}{dx} = \frac{x_{int}y_{int}\left(1-n\right)^2}{\left(x\left(2n-n^2\right) + x_{int}\left(1-n\right)^2\right)^2}$$ **eqn. 5g**

$$-\frac{dy}{dx} = \frac{k}{x_0\sqrt{Q} + x\left(1-\sqrt{Q}\right)}$$ **eqn. 5h**

$$-\frac{dy}{dx} = \frac{ky_0^2}{\left(k\sqrt{Q} + xy_0\left(1-\sqrt{Q}\right)\right)^2}$$ **eqn. 5i**

$$-\frac{dy}{dx} = \frac{Px_0^2}{\left(x\left(1-\sqrt{Q}\right) + x_0\sqrt{Q}\right)^2}$$ **eqn. 5j**

$$-\frac{dy}{dx} = \frac{Py_0^2}{\left(Px\left(1-\sqrt{Q}\right) + y_0\sqrt{Q}\right)^2}$$ **eqn. 5k**

$$-\frac{dy}{dx} = \frac{x_0y_0}{\left(x\left(1-\sqrt{Q}\right) + x_0\sqrt{Q}\right)^2}$$ **eqn. 5l**

$$-\frac{dy}{dx} = \frac{x_{asym}y_{asym}}{Q\left(x - x_{asym}\right)^2}$$ **eqn. 5m**

$$-\frac{dy}{dx} = \frac{x_{int}y_{int}Q}{\left(x\left(1-Q\right) + x_{int}Q\right)^2}$$ **eqn. 5n**

$$-\frac{dy}{dx}=\frac{nx_0y+k\left(1-n\right)^2}{kx_0^2} \quad \textbf{eqn. 6a}$$

$$-\frac{dy}{dx}=\frac{\left(ny+y_0\left(1-n\right)\right)^2}{k} \quad \textbf{eqn. 6b}$$

$$-\frac{dy}{dx}=\frac{\left(P\left(nx_0-x_0\right)-ny\right)^2}{Px_0^2} \quad \textbf{eqn. 6c}$$

$$-\frac{dy}{dx}=\frac{P\left(ny+y_0\left(1-n\right)\right)^2}{y_0^2} \quad \textbf{eqn. 6d}$$

$$-\frac{dy}{dx}=\frac{\left(ny+y_0\left(1-n\right)\right)^2}{x_0y_0} \quad \textbf{eqn. 6e}$$

$$-\frac{dy}{dx}=\frac{\left(1-n\right)^2\left(y-y_{asym}\right)^2}{x_{asym}y_{asym}} \quad \textbf{eqn. 6f}$$

$$-\frac{dy}{dx}=\frac{\left(y\left(2n-n^2\right)+y_{int}\left(1-n\right)^2\right)^2}{x_{int}y_{int}\left(1-n\right)^2} \quad \textbf{eqn. 6g}$$

$$-\frac{dy}{dx}=\frac{\left(k\sqrt{Q}+x_0y\left(1-\sqrt{Q}\right)\right)^2}{kx_0^2} \quad \textbf{eqn. 6h}$$

$$-\frac{dy}{dx}=\frac{\left(y\left(1-\sqrt{Q}\right)+y_0\sqrt{Q}\right)^2}{k} \quad \textbf{eqn. 6i}$$

$$-\frac{dy}{dx}=\frac{\left(Px_0+\left(1-\sqrt{Q}\right)\left(y-Px_0\right)\right)^2}{Px_0^2} \quad \textbf{eqn. 6j}$$

$$-\frac{dy}{dx}=\frac{P\left(y\left(1-\sqrt{Q}\right)+y_0\sqrt{Q}\right)^2}{y_0^2} \quad \textbf{eqn. 6k}$$

$$-\frac{dy}{dx}=\frac{\left(y\left(1-\sqrt{Q}\right)+y_0\sqrt{Q}\right)^2}{x_0y_0} \quad \textbf{eqn. 6l}$$

$$-\frac{dy}{dx}=\frac{Q\left(y-y_{asym}\right)^2}{x_{asym}y_{asym}} \quad \textbf{eqn. 6m}$$

$$-\frac{dy}{dx}=\frac{\left(y\left(1-Q\right)+Qy_{int}\right)^2}{Qx_{int}y_{int}} \quad \textbf{eqn. 6n}$$

The novel invariant function can also be defined with respect to any secant line connecting any two points on its implicit curve, where $\Delta y$ and $\Delta x$ denotes the absolute translocation in the Cartesian plane between any two points on its graph connected by a continuum of intermediate points. The secant line can be defined with respect to either the *x*- or *y*-coordinate (**eqns. 7a-n** and **8a-n**, respectively).

$$-\Delta y = \frac{\Delta x k}{\left(nx + x_0\left(1-n\right)\right)\left(n\left(\Delta x + x\right) + x_0\left(1-n\right)\right)}$$ **eqn. 7a**

$$-\Delta y = \frac{\Delta x k y_0^2}{\left(k\left(1-n\right) + nxy_0\right)\left(k\left(1-n\right) + ny_0\left(\Delta x + x\right)\right)}$$ **eqn. 7b**

$$-\Delta y = \frac{P\Delta x x_0^2}{\left(nx + x_0\left(1-n\right)\right)\left(n\left(\Delta x + x\right) + x_0\left(1-n\right)\right)}$$ **eqn. 7c**

$$-\Delta y = \frac{P\Delta x y_0^2}{\left(Pnx + y_0\left(1-n\right)\right)\left(Pn\left(\Delta x + x\right) + y_0\left(1-n\right)\right)}$$ **eqn. 7d**

$$-\Delta y = \frac{\Delta x x_0 y_0}{\left(nx + x_0\left(1-n\right)\right)\left(n\left(\Delta x + x\right) + x_0\left(1-n\right)\right)}$$ **eqn. 7e**

$$-\Delta y = \frac{\Delta x x_{asym} y_{asym}}{\left(1-n\right)^2\left(x - x_{asym}\right)\left(\Delta x + x - x_{asym}\right)}$$ **eqn. 7f**

$$-\Delta y = \frac{x_{int} y_{int} \Delta x\left(1-n\right)^2}{\left(x\left(2n-n^2\right) + x_{int}\left(1-n\right)^2\right)\left(n\left(2-n\right)\left(x+\Delta x\right) + x_{int}\left(1-n\right)^2\right)}$$ **eqn. 7g**

$$-\Delta y = \frac{\Delta x k}{\left(\left(1-\sqrt{Q}\right)x + x_0\sqrt{Q}\right)\left(\left(1-\sqrt{Q}\right)\left(\Delta x + x\right) + x_0\sqrt{Q}\right)}$$ **eqn. 7h**

$$-\Delta y = \frac{\Delta x k y_0^2}{\left(k\sqrt{Q} + \left(1-\sqrt{Q}\right)xy_0\right)\left(k\sqrt{Q} + y_0\left(1-\sqrt{Q}\right)\left(\Delta x + x\right)\right)}$$ **eqn. 7i**

$$-\Delta y = \frac{P\Delta x x_0^2}{\left(\left(1-\sqrt{Q}\right)x + x_0\sqrt{Q}\right)\left(\left(1-\sqrt{Q}\right)\left(\Delta x + x\right) + x_0\sqrt{Q}\right)}$$ **eqn. 7j**

$$-\Delta y = \frac{P\Delta x y_0^2}{\left(P\left(1-\sqrt{Q}\right)x + y_0\sqrt{Q}\right)\left(P\left(1-\sqrt{Q}\right)\left(\Delta x + x\right) + y_0\sqrt{Q}\right)}$$ **eqn. 7k**

$$-\Delta y = \frac{\Delta x x_0 y_0}{\left(\left(1-\sqrt{Q}\right)x + x_0\sqrt{Q}\right)\left(\left(1-\sqrt{Q}\right)\left(\Delta x + x\right) + x_0\sqrt{Q}\right)}$$ **eqn. 7l**

$$-\Delta y = \frac{\Delta x x_{asym} y_{asym}}{Q\left(x - x_{asym}\right)\left(\Delta x + x - x_{asym}\right)}$$ **eqn. 7m**

$$-\Delta y = \frac{Q\Delta x x_{int} y_{int}}{\left(Qx_{int} + x\left(1-Q\right)\right)\left(Qx_{int} + \left(1-Q\right)\left(x + \Delta x\right)\right)}$$ **eqn. 7n**

$$-\Delta y = \frac{\Delta x \left(k\left(1-n\right)+nx_0y\right)^2}{x_0\left(\Delta xn\left(k\left(1-n\right)+nx_0y\right)+kx_0\right)}$$ **eqn. 8a**

$$-\Delta y = \frac{\Delta x \left(ny+y_0\left(1-n\right)\right)^2}{\Delta xn\left(ny+y_0\left(1-n\right)\right)+k}$$ **eqn. 8b**

$$-\Delta y = \frac{\Delta x \left(Px_0\left(1-n\right)+ny\right)^2}{Px_0^2+\Delta xn\left(Px_0\left(1-n\right)+ny\right)}$$ **eqn. 8c**

$$-\Delta y = \frac{P\Delta x \left(ny+y_0\left(1-n\right)\right)^2}{P\Delta xn\left(ny+y_0\left(1-n\right)\right)+y_0^2}$$ **eqn. 8d**

$$-\Delta y = \frac{\Delta x \left(ny+y_0\left(1-n\right)\right)^2}{\Delta xn\left(ny+y_0\left(1-n\right)\right)+x_0y_0}$$ **eqn. 8e**

$$-\Delta y = \frac{\Delta x \left(1-n\right)^2\left(y-y_{asym}\right)^2}{\Delta x\left(1-n\right)^2\left(y-y_{asym}\right)+x_{asym}y_{asym}}$$ **eqn. 8f**

$$-\Delta y = \frac{\Delta x \left(y\left(2n-n^2\right)+y_{int}\left(1-n\right)^2\right)^2}{n^2y\Delta x\left(2-n\right)^2+y_{int}\left(1-n\right)^2\left(x_{int}+\Delta x\left(2n-n^2\right)\right)}$$ **eqn. 8g**

$$-\Delta y = \frac{\Delta x \left(k\sqrt{Q}+x_0y\left(1-\sqrt{Q}\right)\right)^2}{x_0\left(\Delta x\left(1-\sqrt{Q}\right)\left(k\sqrt{Q}+x_0y\left(1-\sqrt{Q}\right)\right)+kx_0\right)}$$ **eqn. 8h**

$$-\Delta y = \frac{\Delta x \left(\left(1-\sqrt{Q}\right)y+y_0\sqrt{Q}\right)^2}{\Delta x\left(1-\sqrt{Q}\right)\left(y\left(1-\sqrt{Q}\right)+y_0\sqrt{Q}\right)+k}$$ **eqn. 8i**

$$-\Delta y = \frac{\Delta x \left(Px_0\sqrt{Q}+\left(1-\sqrt{Q}\right)y\right)^2}{Px_0^2+\Delta x\left(1-\sqrt{Q}\right)\left(Px_0\sqrt{Q}+y\left(1-\sqrt{Q}\right)\right)}$$ **eqn. 8j**

$$-\Delta y = \frac{P\Delta x \left(y\left(1-\sqrt{Q}\right)+y_0\sqrt{Q}\right)^2}{P\Delta x\left(1-\sqrt{Q}\right)\left(y\left(1-\sqrt{Q}\right)+y_0\sqrt{Q}\right)+y_0^2}$$ **eqn. 8k**

$$-\Delta y = \frac{\Delta x \left(y\left(1-\sqrt{Q}\right)+y_0\sqrt{Q}\right)^2}{\Delta x\left(1-\sqrt{Q}\right)\left(y\left(1-\sqrt{Q}\right)+y_0\sqrt{Q}\right)+x_0y_0}$$ **eqn. 8l**

$$-\Delta y = \frac{\Delta xQ\left(y-y_{asym}\right)^2}{\Delta xQ\left(y-y_{asym}\right)+x_{asym}y_{asym}}$$ **eqn. 8m**

$$-\Delta y = \frac{\Delta x \left(Qy_{int}+y\left(1-Q\right)\right)^2}{Qy_{int}\left(x_{int}+\Delta x\left(1-Q\right)\right)+y\Delta x\left(1-Q\right)^2}$$ **eqn. 8n**

The novel invariant function can also be defined with respect to the gradient of said secant lines, denoted here as $\Delta y\ /\Delta x$, and can be defined with respect to either the $x$- or $y$-coordinate (**eqns. 9a-n** and **10a-n**). As $\Delta x$ approaches zero, $\Delta y/\Delta x$ approaches the instantaneous rate of change at any point on its implicit curve and forms an equality with **eqns. 5a-n** and **6a-n**.

$$-\frac{\Delta y}{\Delta x} = \frac{k}{\left(nx + x_0\left(1-n\right)\right)\left(n\left(\Delta x + x\right) + x_0\left(1-n\right)\right)} \quad \textbf{eqn. 9a}$$

$$-\frac{\Delta y}{\Delta x} = \frac{k y_0^2}{\left(k\left(1-n\right) + nxy_0\right)\left(k\left(1-n\right) + ny_0\left(\Delta x + x\right)\right)} \quad \textbf{eqn. 9b}$$

$$-\frac{\Delta y}{\Delta x} = \frac{P x_0^2}{\left(nx + x_0\left(1-n\right)\right)\left(n\left(\Delta x + x\right) + x_0\left(1-n\right)\right)} \quad \textbf{eqn. 9c}$$

$$-\frac{\Delta y}{\Delta x} = \frac{P y_0^2}{\left(Pnx + y_0\left(1-n\right)\right)\left(Pn\left(\Delta x + x\right) + y_0\left(1-n\right)\right)} \quad \textbf{eqn. 9d}$$

$$-\frac{\Delta y}{\Delta x} = \frac{x_0 y_0}{\left(nx + x_0\left(1-n\right)\right)\left(n\left(\Delta x + x\right) + x_0\left(1-n\right)\right)} \quad \textbf{eqn. 9e}$$

$$-\frac{\Delta y}{\Delta x} = \frac{x_{asym} y_{asym}}{\left(1-n\right)^2\left(x - x_{asym}\right)\left(\Delta x + x - x_{asym}\right)} \quad \textbf{eqn. 9f}$$

$$-\frac{\Delta y}{\Delta x} = \frac{x_{int} y_{int}\left(1-n\right)^2}{\left(x\left(2n-n^2\right) + x_{int}\left(1-n\right)^2\right)\left(\left(2n-n^2\right)\left(x + \Delta x\right) + x_{int}\left(1-n\right)^2\right)} \quad \textbf{eqn. 9g}$$

$$-\frac{\Delta y}{\Delta x} = \frac{k}{\left(x\left(1-\sqrt{Q}\right) + x_0\sqrt{Q}\right)\left(\left(1-\sqrt{Q}\right)\left(\Delta x + x\right) + x_0\sqrt{Q}\right)} \quad \textbf{eqn. 9h}$$

$$-\frac{\Delta y}{\Delta x} = \frac{k y_0^2}{\left(k\sqrt{Q} + xy_0\left(1-\sqrt{Q}\right)\right)\left(k\sqrt{Q} + y_0\left(1-\sqrt{Q}\right)\left(\Delta x + x\right)\right)} \quad \textbf{eqn. 9i}$$

$$-\frac{\Delta y}{\Delta x} = \frac{P x_0^2}{\left(x\left(1-\sqrt{Q}\right) + x_0\sqrt{Q}\right)\left(\left(1-\sqrt{Q}\right)\left(\Delta x + x\right) + x_0\sqrt{Q}\right)} \quad \textbf{eqn. 9j}$$

$$-\frac{\Delta y}{\Delta x} = \frac{P y_0^2}{\left(Px\left(1-\sqrt{Q}\right) + y_0\sqrt{Q}\right)\left(P\left(1-\sqrt{Q}\right)\left(\Delta x + x\right) + y_0\sqrt{Q}\right)} \quad \textbf{eqn. 9k}$$

$$-\frac{\Delta y}{\Delta x} = \frac{x_0 y_0}{\left(x\left(1-\sqrt{Q}\right) + x_0\sqrt{Q}\right)\left(\left(1-\sqrt{Q}\right)\left(\Delta x + x\right) + x_0\sqrt{Q}\right)} \quad \textbf{eqn. 9l}$$

$$-\frac{\Delta y}{\Delta x} = \frac{x_{asym} y_{asym}}{Q\left(x - x_{asym}\right)\left(\Delta x + x - x_{asym}\right)} \quad \textbf{eqn. 9m}$$

$$-\frac{\Delta y}{\Delta x} = \frac{x_{int} y_{int} Q}{\left(x\left(1-Q\right) + x_{int}Q\right)\left(\left(1-Q\right)\left(x + \Delta x\right) + x_{int}Q\right)} \quad \textbf{eqn. 9n}$$

$$-\frac{\Delta y}{\Delta x}=\frac{\left(k\left(1-n\right)+nx_0y\right)^2}{x_0\left(\Delta xn\left(k\left(1-n\right)+nx_0y\right)+kx_0\right)}$$ **eqn. 10a**

$$-\frac{\Delta y}{\Delta x}=\frac{\left(ny+y_0\left(1-n\right)\right)^2}{\Delta xn\left(ny+y_0\left(1-n\right)\right)+k}$$ **eqn. 10b**

$$-\frac{\Delta y}{\Delta x}=\frac{\left(Px_0\left(1-n\right)+ny\right)^2}{Px_0^2+\Delta xn\left(Px_0\left(1-n\right)+ny\right)}$$ **eqn. 10c**

$$-\frac{\Delta y}{\Delta x}=\frac{P\left(ny+y_0\left(1-n\right)\right)^2}{P\Delta xn\left(ny+y_0\left(1-n\right)\right)+y_0^2}$$ **eqn. 10d**

$$-\frac{\Delta y}{\Delta x}=\frac{\left(ny+y_0\left(1-n\right)\right)^2}{\Delta xn\left(ny+y_0\left(1-n\right)\right)+x_0y_0}$$ **eqn. 10e**

$$-\frac{\Delta y}{\Delta x}=\frac{\left(1-n\right)^2\left(y-y_{asym}\right)^2}{\Delta x\left(1-n\right)^2\left(y-y_{asym}\right)+x_{asym}y_{asym}}$$ **eqn. 10f**

$$-\frac{\Delta y}{\Delta x}=\frac{\left(y\left(2n-n^2\right)+y_{int}\left(1-n\right)^2\right)^2}{n^2y\Delta x\left(2-n\right)^2+y_{int}\left(1-n\right)^2\left(x_{int}+\Delta x\left(2n-n^2\right)\right)}$$ **eqn. 10g**

$$-\frac{\Delta y}{\Delta x}=\frac{\left(k\sqrt{Q}+x_0y\left(1-\sqrt{Q}\right)\right)^2}{x_0\left(\Delta x\left(1-\sqrt{Q}\right)\left(k\sqrt{Q}+x_0y\left(1-\sqrt{Q}\right)\right)+kx_0\right)}$$ **eqn. 10h**

$$-\frac{\Delta y}{\Delta x}=\frac{\left(y\left(1-\sqrt{Q}\right)+y_0\sqrt{Q}\right)^2}{\Delta x\left(1-\sqrt{Q}\right)\left(y\left(1-\sqrt{Q}\right)+y_0\sqrt{Q}\right)+k}$$ **eqn. 10i**

$$-\frac{\Delta y}{\Delta x}=\frac{\left(Px_0\sqrt{Q}+y\left(1-\sqrt{Q}\right)\right)^2}{Px_0^2+\Delta x\left(1-\sqrt{Q}\right)\left(Px_0\sqrt{Q}+y\left(1-\sqrt{Q}\right)\right)}$$ **eqn. 10j**

$$-\frac{\Delta y}{\Delta x}=\frac{P\left(y\left(1-\sqrt{Q}\right)+y_0\sqrt{Q}\right)^2}{P\Delta x\left(1-\sqrt{Q}\right)\left(y\left(1-\sqrt{Q}\right)+y_0\sqrt{Q}\right)+y_0^2}$$ **eqn. 10k**

$$-\frac{\Delta y}{\Delta x}=\frac{\left(y\left(1-\sqrt{Q}\right)+y_0\sqrt{Q}\right)^2}{\Delta x\left(1-\sqrt{Q}\right)\left(y\left(1-\sqrt{Q}\right)+y_0\left(1-n\right)\right)+x_0y_0}$$ **eqn. 10l**

$$-\frac{\Delta y}{\Delta x}=\frac{Q\left(y-y_{asym}\right)^2}{\Delta xQ\left(y-y_{asym}\right)+x_{asym}y_{asym}}$$ **eqn. 10m**

$$-\frac{\Delta y}{\Delta x}=\frac{\left(Qy_{int}+y\left(1-Q\right)\right)^2}{Qy_{int}\left(x_{int}+\Delta x\left(1-Q\right)\right)+y\Delta x\left(1-Q\right)^2}$$ **eqn. 10n**

The equations depicted in the sections above are non-exhaustive. Each parameter belongs to a set and has an explicit relationship to other parameters within its own set, and between sets of other parameters. These relationships include important equivalences between fundamental parameters, and explicit pairs of other parameters, including but are not necessarily limited to those defined in **eqns. 11a-d** and **12**.

$$k = x_0 y_0 = P x_0^2 = \frac{y_0^2}{P}$$ **eqn. 11a**

$$P = \sqrt{P_a P_b} = \frac{y_0}{x_0} = \frac{k}{x_0^2} = \frac{y_0^2}{k} = \frac{y_{asym}}{x_{asym}} = \frac{y_{int}}{x_{int}}$$ **eqn. 11b**

$$Q = \sqrt{\frac{P_b}{P_a}} = \frac{P}{P_a} = \frac{P_b}{P} = \frac{y_{asym}}{y_{asym} - y_{int}} = \frac{x_{asym}}{x_{asym} - x_{int}} = (1-n)^2$$ **eqn. 11c**

$$n = 1 - \sqrt[4]{\frac{P_b}{P_a}} = \frac{y_0}{y_0 - y_{asym}} = \frac{x_0}{x_0 - x_{asym}} = \frac{2y_0 - y_{int}}{y_0 - y_{int}} = \frac{2x_0 - x_{int}}{x_0 - x_{int}}$$ **eqn. 11d**

The parameter *n* is redundant with the parameter *u*; the two are the reciprocal of each other (**eqn. 12**).

$$nu = 1$$ **eqn. 12**

From the established theory, an expanded network of algebraic identities can be defined. The totality of the geometric relationships is too vast to list in its entirety; however, a representative set is provided with **eqns. 13a-s**.

$$P = \frac{y_{asym} - y_0}{x_{asym} - x_0} = \frac{y_{asym} + y_0}{x_{asym} + x_0} = \frac{y_{int} - y_0}{x_{int} - x_0} = \frac{y_{int} + y_0}{x_{int} + x_0}$$ **eqn. 13a**

$$P_a = \frac{k}{x_0^2 (1-n)^2} = \frac{y_0^2}{k (1-n)^2} = \frac{P}{(1-n)^2} = \frac{y_0}{x_0 (1-n)^2} = \frac{y_{asym}}{x_{asym} (1-n)^2}$$ **eqn. 13b**

$$P_a = \frac{y_0 (x_0 - x_{int})^2}{x_0^3} = \frac{(y_0 - y_{int})^2}{k} = \frac{k}{\left(n x_{int} - \sqrt{\frac{k}{P_b}}\right)^2} = \frac{\left(n y_{int} + \sqrt{P_b k}\right)^2}{k}$$ **eqn. 13c**

$$P_b = \frac{k (1-n)^2}{x_0^2} = \frac{y_0^2 (1-n)^2}{k} = P (1-n)^2 = \frac{y_0 (1-n)^2}{x_0} = \frac{y_{asym} (1-n)^2}{x_{asym}}$$ **eqn. 13d**

$$P_b = \frac{y_0^3}{x_0 (y_0 - y_{int})^2} = \frac{k}{(x_0 - x_{int})^2} = \frac{k}{\left(n x_{int} + \sqrt{\frac{k}{P_a}}\right)^2} = \frac{\left(n y_{int} - \sqrt{P_a k}\right)^2}{k}$$ **eqn. 13e**

$$y_{int} = \frac{k (2-n)}{x_0 (1-n)} = \frac{y_0 (2-n)}{1-n} = \frac{P x_0 (2-n)}{1-n} = y_{asym} - \frac{y_{asym}}{(n-1)^2}$$ **eqn. 13f**

$$x_{int} = \frac{k (2-n)}{y_0 (1-n)} = \frac{x_0 (2-n)}{1-n} = \frac{y_0 (2-n)}{P (1-n)} = x_{asym} - \frac{x_{asym}}{(1-n)^2}$$ **eqn. 13g**

$$k = P_a x_0^2 (1-n)^2 = \frac{y_0^2}{P_a (1-n)^2} = \frac{P_b x_0^2}{(1-n)^2} = \frac{y_0^2 (1-n)^2}{P_b}$$ **eqn. 13h**

$$k = \frac{x_0 y_{int}(1-n)}{2-n} = \frac{x_{int} y_0 (1-n)}{2-n} = \frac{y_{int}^2 (1-n)^2}{P(2-n)^2} = \frac{P x_{int}^2 (1-n)^2}{(2-n)^2}$$ **eqn. 13i**

$$k = \frac{x_{int} y_{int} (1-n)^2}{(2-n)^2} = \frac{n^2 x_{asym} y_{asym}}{(1-n)^2}$$ **eqn. 13j**

$$x_0 = \frac{\sqrt{\frac{k}{P_a}}}{(1-n)} = \sqrt{\frac{k}{P_b}}(1-n) = \frac{y_0}{P_a (1-n)^2} = \frac{y_0 (1-n)^2}{P_b}$$ **eqn. 13k**

$$x_0 = \frac{k(2-n)}{y_{int}(1-n)} = \frac{y_{int}(1-n)}{P(2-n)} = \frac{x_{int}(1-n)}{2-n} = \frac{x_{asym} y_0}{y_{asym}}$$ **eqn. 13l**

$$y_0 = \sqrt{P_a k}(1-n) = \frac{\sqrt{P_b k}}{1-n} = P_a x_0 (1-n)^2 = \frac{P_b x_0}{(1-n)^2}$$ **eqn. 13m**

$$y_0 = \frac{y_{int}(1-n)}{2-n} = \frac{k(2-n)}{x_{int}(1-n)} = \frac{P x_{int}(1-n)}{2-n} = \frac{y_{asym} x_0}{x_{asym}}$$ **eqn. 13n**

$$y_{asym} = \frac{y_{int}(1-n)^2}{(1-n)^2 - 1} = \frac{P x_{int}(1-n)^2}{(1-n)^2 - 1} = \frac{y_0 (n-1)}{n}$$ **eqn. 13o**

$$x_{asym} = \frac{x_{int}(1-n)^2}{(1-n)^2 - 1} = \frac{y_{int}(1-n)^2}{P\left((1-n)^2 - 1\right)} = \frac{x_0 (n-1)}{n}$$ **eqn. 13p**

$$(1-n)^2 = Q = \frac{x_{asym} y_{asym} - \sqrt{x_{asym} y_{asym}}\sqrt{x_{int} y_{int}}}{x_{asym} y_{asym} - x_{int} y_{int}}$$ **eqn. 13q**

$$1 - n = \sqrt{Q} = \frac{k}{\sqrt{x_{int} y_{int}} - \sqrt{k}}$$ **eqn. 13r**

$$n = 1 - \sqrt{Q} = \frac{\sqrt{k}\left(\sqrt{P_a} - \sqrt{P_b}\right)}{y_{int}} = \frac{\sqrt{k}\left(\sqrt{\frac{1}{P_b}} - \sqrt{\frac{1}{P_a}}\right)}{x_{int}}$$ **eqn. 13s**

The equalities presented above are summarized in **eqns. 14a-c** and give precise identities for the fundamental parameters $n$, $P$, $Q$, $k$, $x_{asym}$, $y_{asym}$, $x_{int}$, and $y_{int}$ with respect to each other.

$$n^2 = \left(1 - \sqrt{Q}\right)^2 = \frac{k(P_a + P_b - 2P)}{P x_{int} y_{int}}$$ **eqn. 14a**

$$\frac{(1-n)^4}{n^4 (2-n)^2} = \frac{Q^2}{(1-Q)^4 (1+Q)^2} = \frac{P x_{asym} y_{asym}}{k(P_a + P_b - 2P)}$$ **eqn. 14b**

$$\frac{(1-n)^2}{n^2 (2-n)^2} = \frac{Q}{(1-Q)^2} = \frac{P}{P_a + P_b - 2P}$$ **eqn. 14c**

The parameters algebraically defined in the above section is now elaborated in the context of their geometric meaning. The descriptions provided here are for illustrative and explanatory purposes only, and should not be conflated with any use case, or be considered a constraint, or boundary in any manner to the generality of the theory disclosed thus far or discussed in the later sections of this document.

The novel invariant function is characterized as having a point on its implicit curve with coordinates $x = x_0$, $y = y_0$, which coincide with the graph of **eqns. 1a-c** (**Figure 1a-c**). The slope of the tangent line at this point is $-P$ and is equal in both implicit curves of the novel function and **eqns. 1d-e** (**Figure 2a-c**). The parameter $k$ is the hyperbola constant of **eqns. 1a-c**, (i.e $xy = \underline{k}$), and denotes the magnitude of the constant product's implicit curve with which the implicit curve of the novel invariant function shares the coordinates $x = x_0$, $y = y_0$ (**Figures 3a-g**). The parameters $x_{\text{asym}}$ and $y_{\text{asym}}$ denote the "vertical" and "horizontal" asymptotes, respectively, of the novel function's implicit curve (**Figures 4a-l**). However, the designations "vertical" and/or "horizontal" should not be misconstrued as perpendicular lines, or straight lines in all embodiments of the disclosed invention; these terms are used only for ease of communication, and the divergence of their meaning from the status quo is discussed elsewhere in this document. The parameters $x_{\text{int}}$ and $y_{\text{int}}$ denote the $x$- and $y$-intercepts the novel function's implicit curve (**Figure 5a-g**). The slope of the tangent lines at the $y$- and $x$-intercepts of the novel implicit curve, $y = y_{\text{int}}$ and $x = x_{\text{int}}$, respectively, are denoted by the parameters $P_a$ and $P_b$, respectively (**Figure 6a-f**).

In one embodiment, the parameters $n$, $u$ denote the relative size of two rectangles, for which there are several methods of discovery. One method is described here. Assume both rectangles have edges parallel to the $x$- and $y$-axis. Let the first of the two rectangles have one corner at the origin ($x = 0$, $y = 0$), one corner at the $x$-intercept ($y = 0$, $x = x_{\text{int}}$), one corner at the $y$-intercept ($x = 0$, $y = y_{\text{int}}$), and one corner at the junction of the lines emanating from both intercepts ($x = x_{\text{int}}$, $y = y_{\text{int}}$) and converging at a right angle. The second rectangle has one corner at the point on the implicit curve of **eqns. 1a-c** where a tangent line at this point is parallel to $P_b$ $\left(x = \sqrt{k}/\sqrt{P_b}\,, y = \sqrt{kP_b}\right)$, and one corner at the point on the implicit curve of **eqns. 1a-c** where a tangent line at this point is parallel to $P_a$ $\left(x = \sqrt{k}/\sqrt{P_a}\,, y = \sqrt{kP_a}\right)$, and the other corners at the junctions of the lines emanating from both other points and converging at right angles (**Figure 7a-b**). The square root of the quotient of the rectangles' areas, or the quotient of the rectangles' lengths, or the quotient of the rectangles' heights, can be used to calculate to $n$ and its reciprocal, $u$.

In one embodiment, the parameter $Q$ also denotes the relative size of two rectangles, which may be discovered through several geometric processes; one method is described here. Assume both rectangles have edges parallel to the $x$- and $y$-axis. Let both rectangles have one corner at junction of the horizontal and vertical asymptotes of the implicit curve of the novel invariant function ($x = x_{\text{asym}}$, $y = y_{\text{asym}}$). Then, let the first of the two rectangles have a corner at the convergence of the lines emanating from the $x$- and $y$-intercepts ($x = x_{\text{int}}$, $y = y_{\text{int}}$), and the other corners at the junctions of the lines emanating from both other points and converging at right angles. The second rectangle has one corner at the origin ($x = 0$, $y = 0$), and the other corners at the junctions of the lines emanating from both other points and converging at right angles (**Figure 7c-d**). The quotient of the square root of the area of the one rectangle to the square root of the sum of the areas of both rectangles, or the quotient of the length of one rectangle to the sum of the lengths of both rectangles, or the quotient of the height of one rectangle to the sum of the heights of both rectangles, can be used to calculate $Q$. The parameters $n$, $u$ and $Q$ report the relative scaling of the transformation of the novel invariant function's implicit curve with respect to the constant product implicit curve, and vice versa.

The dimensions of the graphs in the following figures hold no special significance and are chosen only for the sake of demonstration; the disclosed novel function is effectively unbounded and can be applied to any plane or hyperplane, regardless of its size.

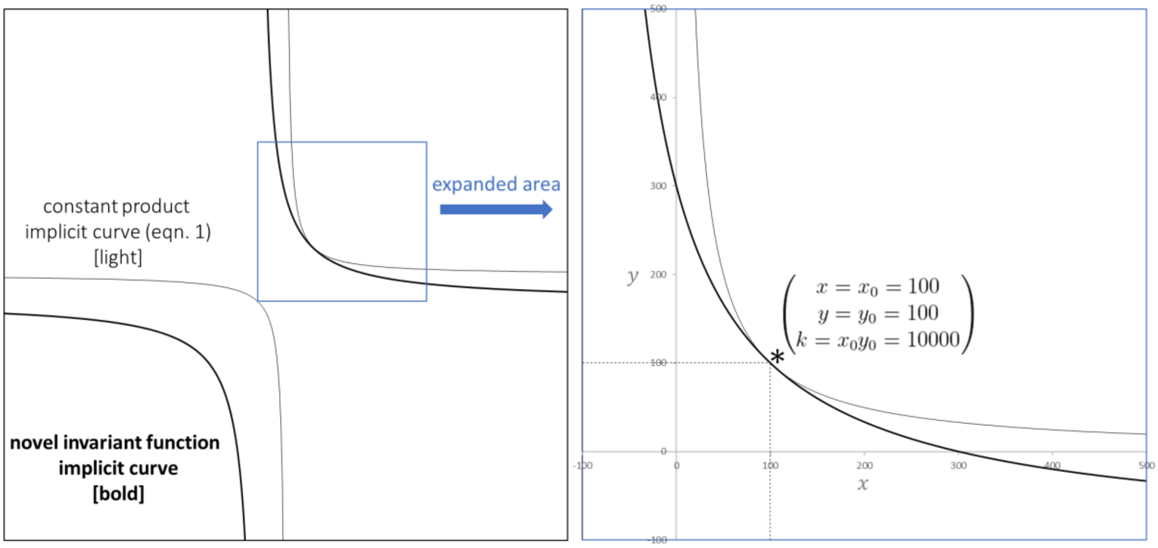



**Figure 1a.**

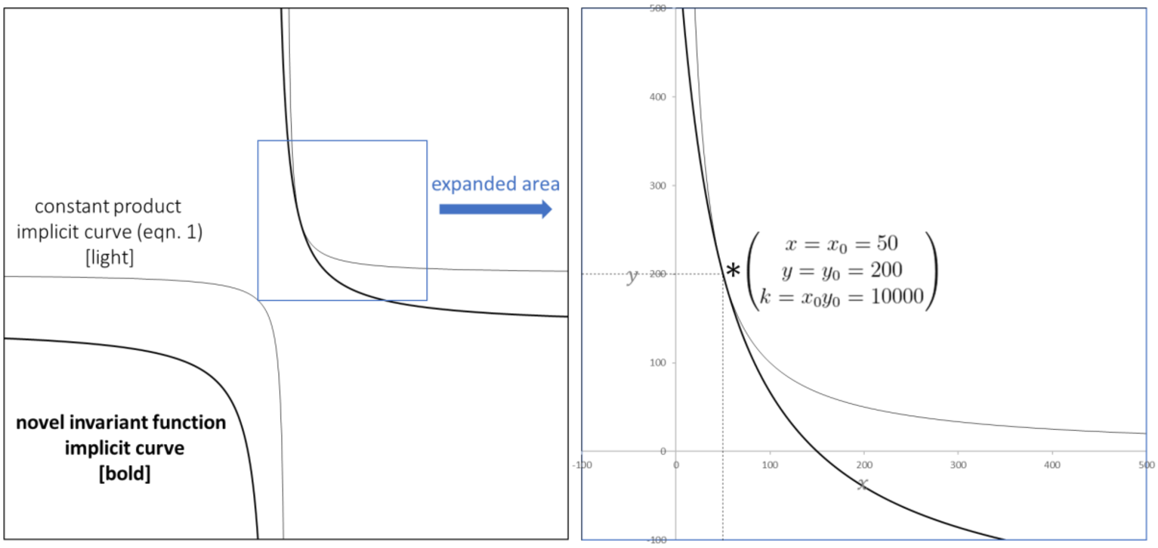



**Figure 1b.**

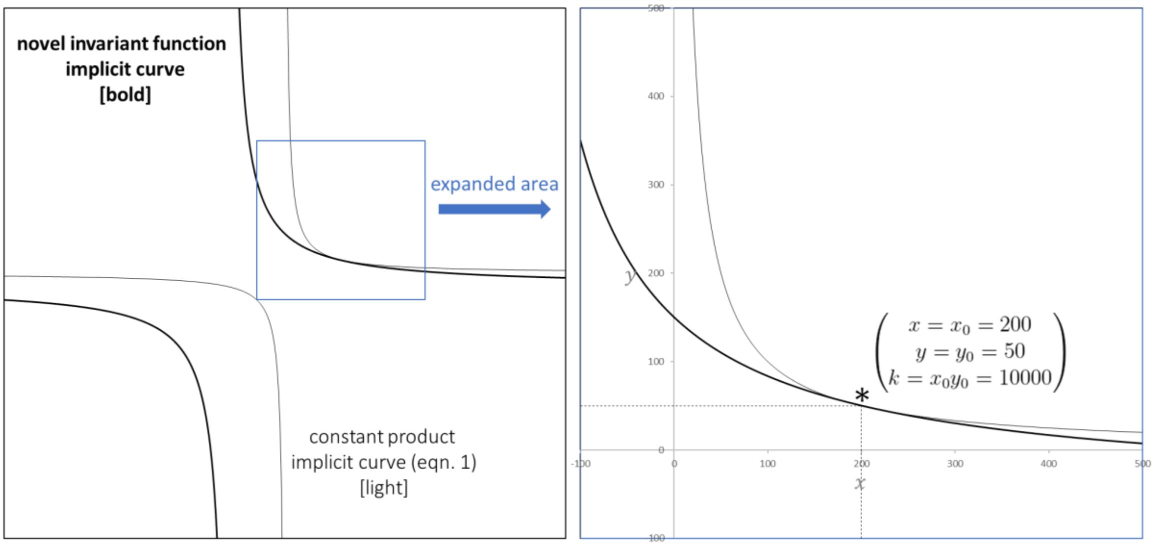

novel invariant function
implicit curve
[bold]
expanded area
constant product
implicit curve (eqn. 1)
[light]
$x = x_0 = 200$
$y = y_0 = 50$
$k = x_0 y_0 = 10000$
y
x

**Figure 1c.**

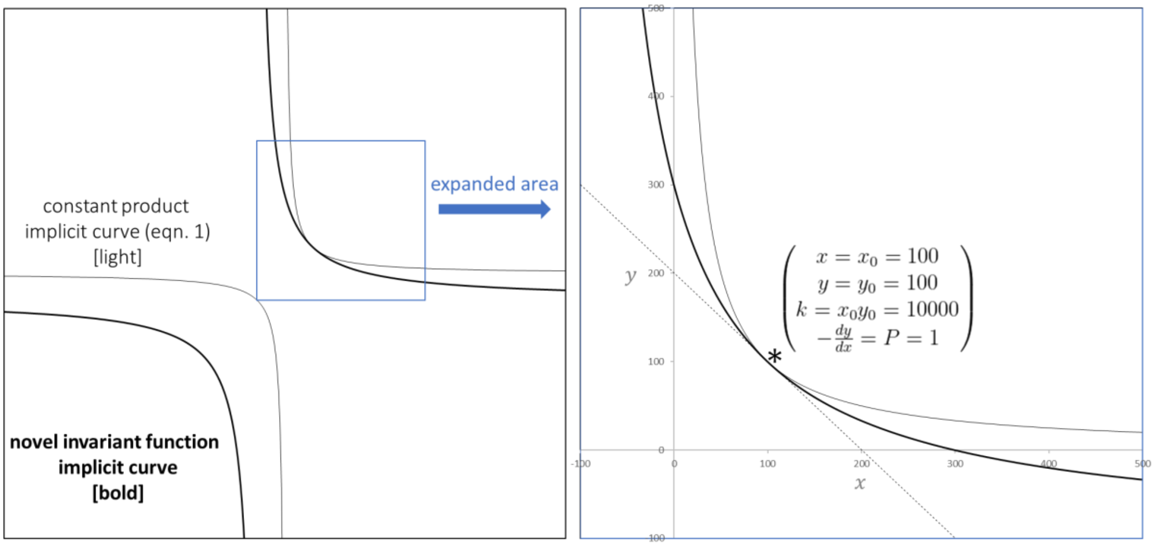



**Figure 2a.**

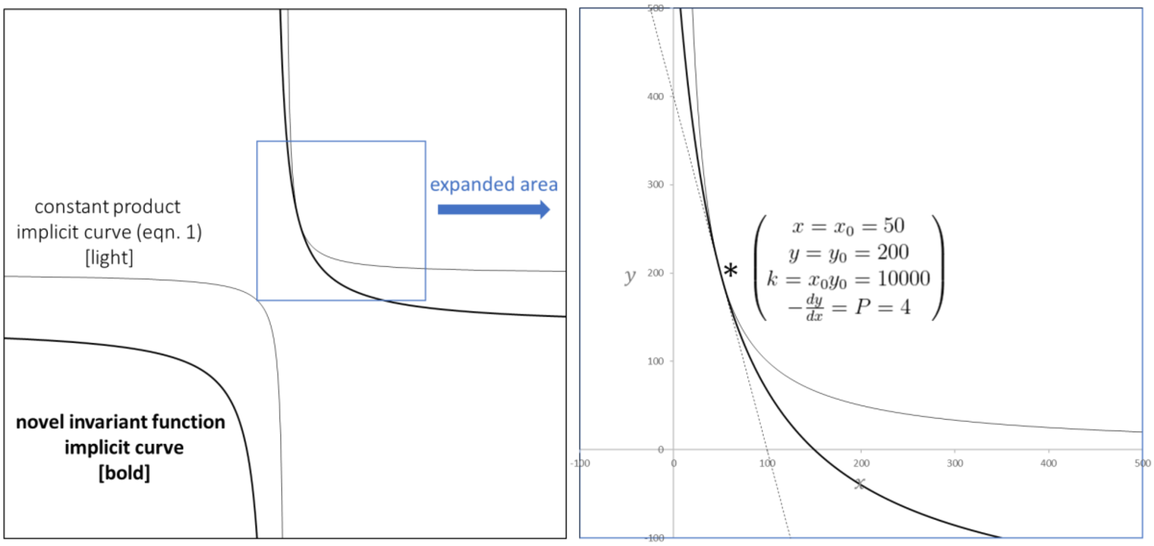



**Figure 2b.**

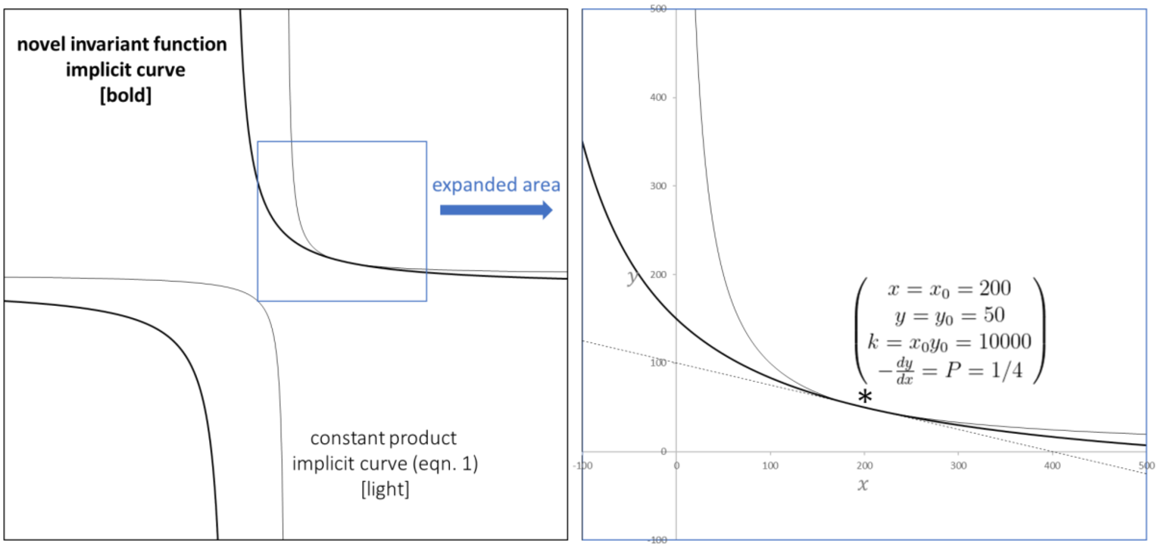

novel invariant function
implicit curve
[bold]
expanded area
constant product
implicit curve (eqn. 1)
[light]
$x = x_0 = 200$
$y = y_0 = 50$
$k = x_0 y_0 = 10000$
$-\frac{dy}{dx} = P = 1/4$
y
x

**Figure 2c.**

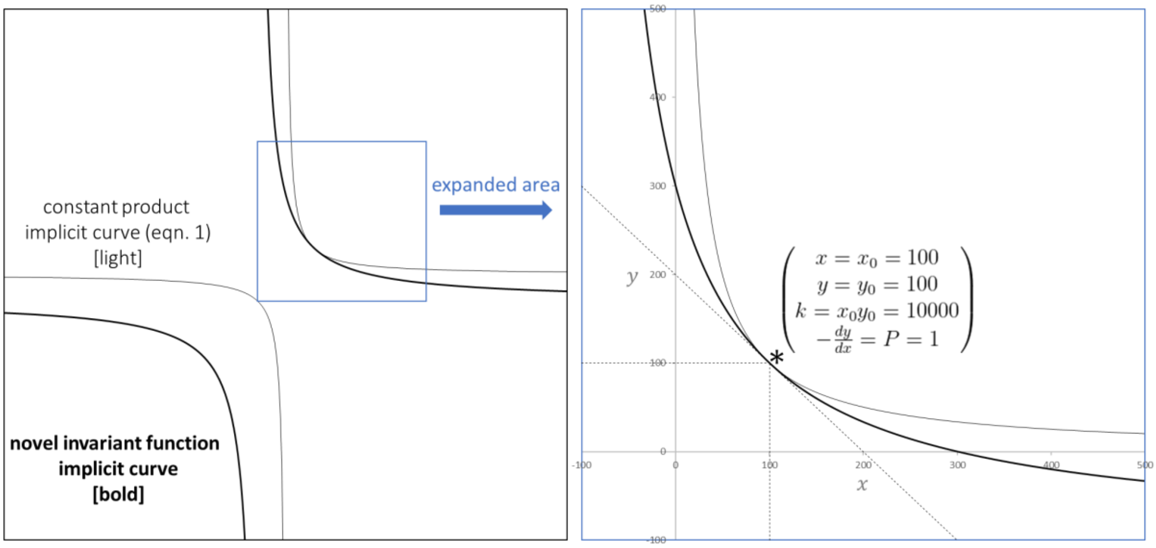



**Figure 3a.**

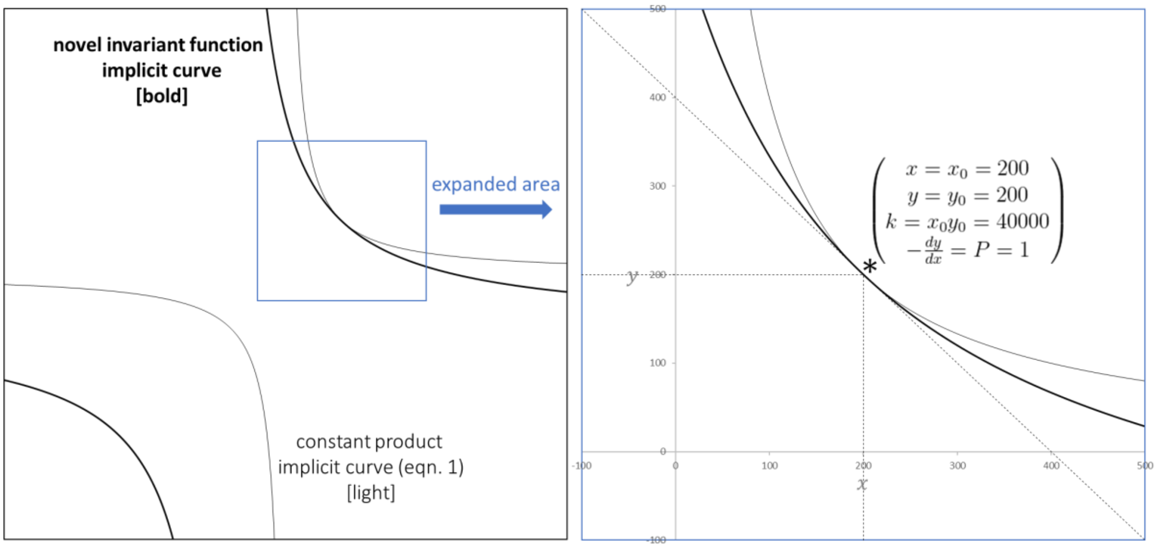



**Figure 3b.**

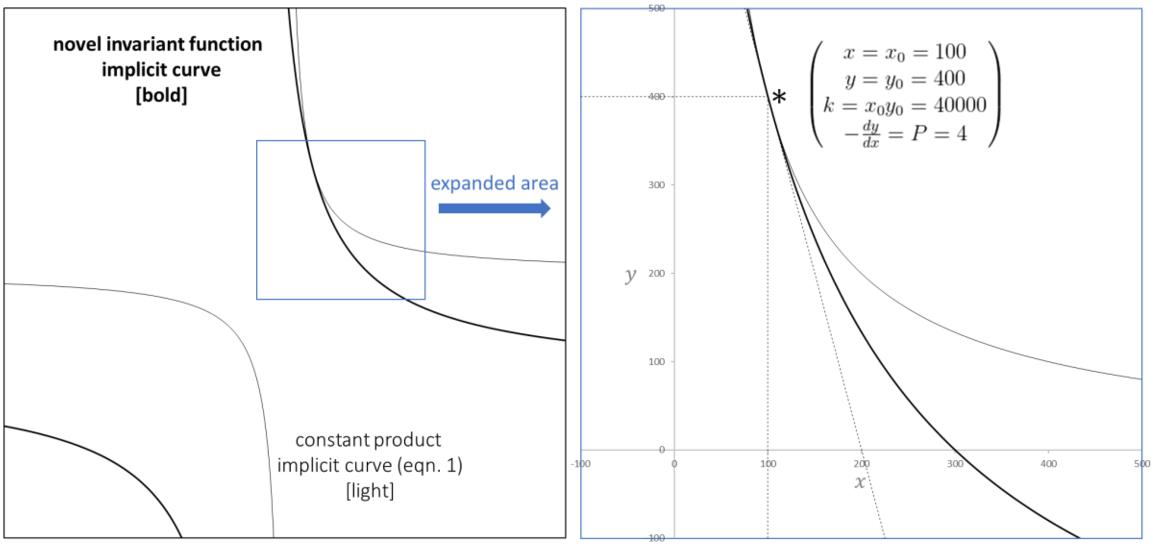



**Figure 3c.**

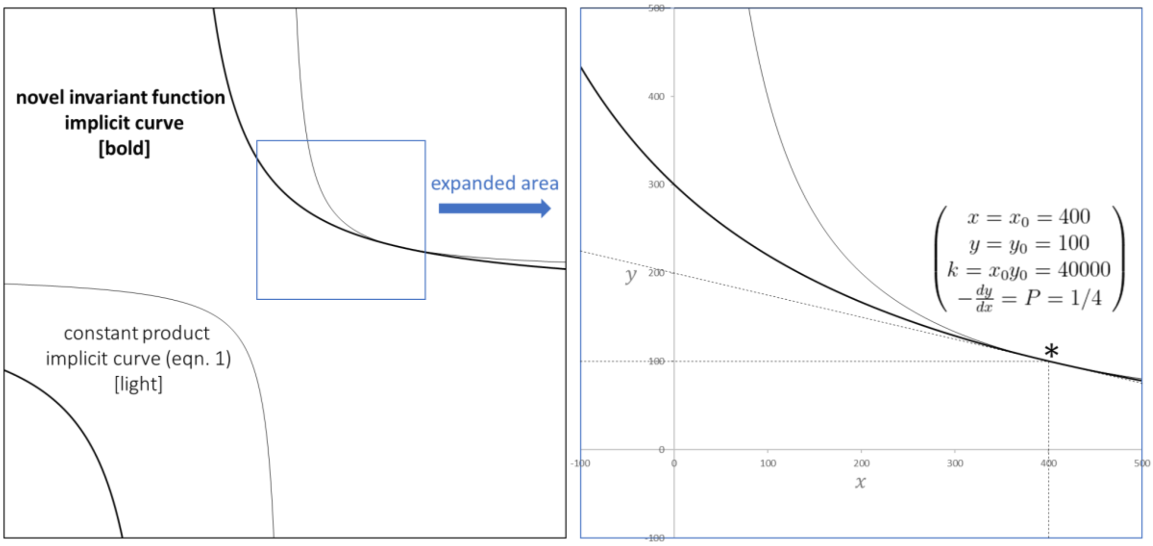



**Figure 3d.**

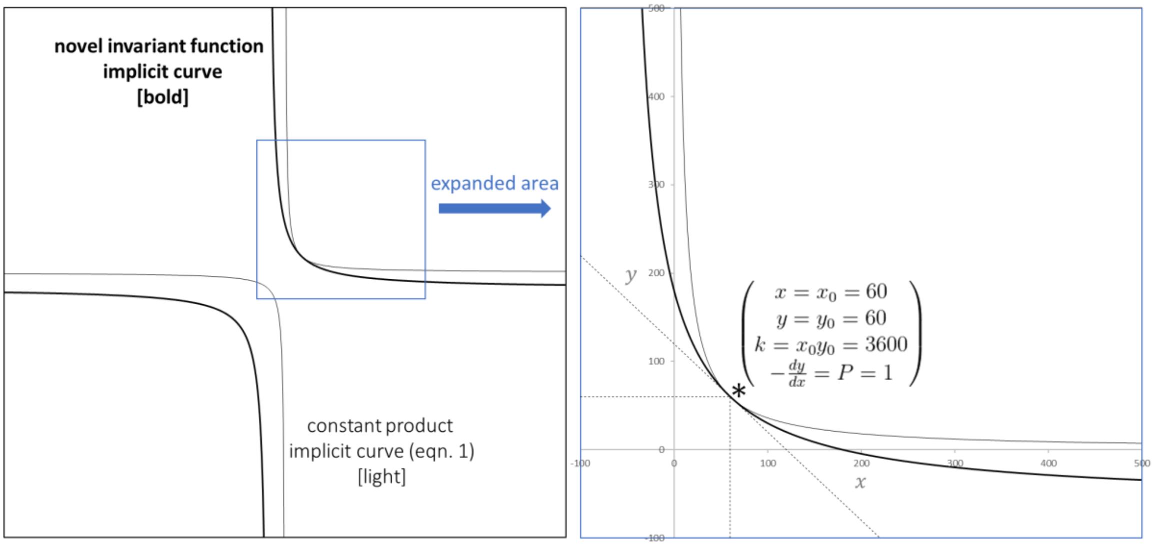



**Figure 3e.**

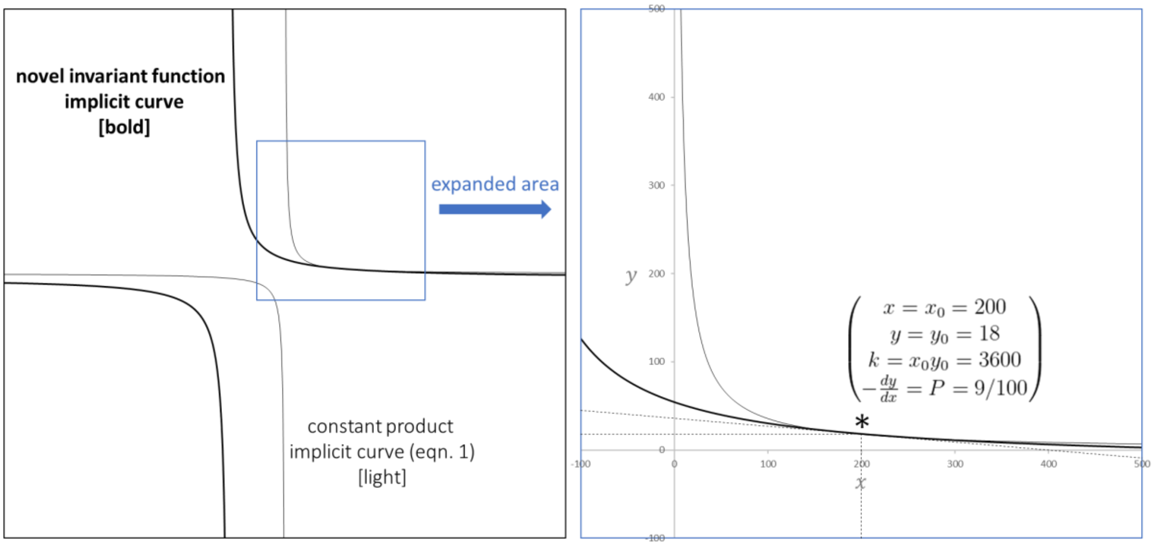



**Figure 3f.**

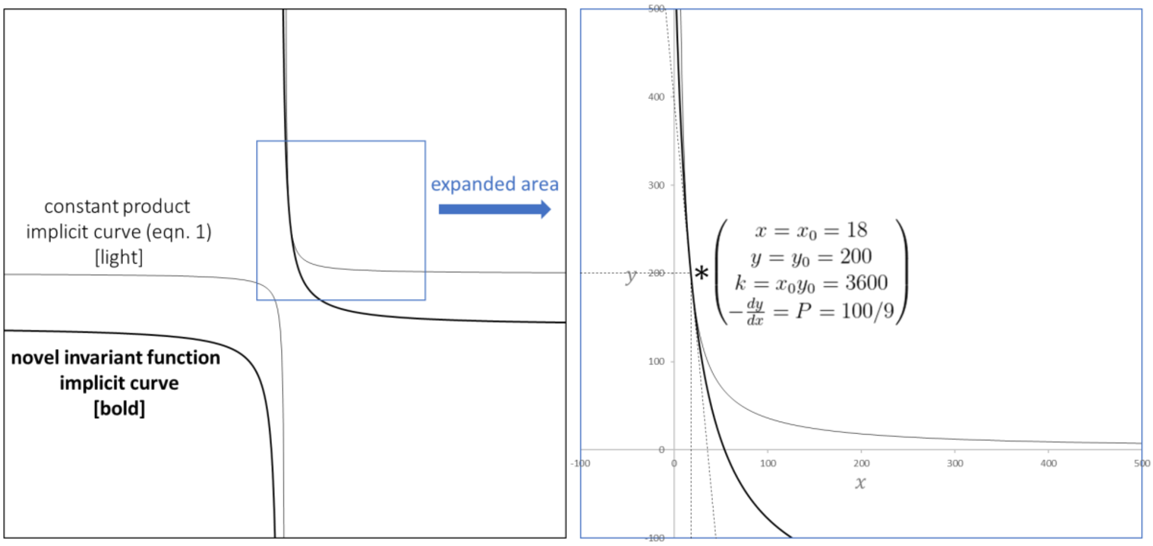

constant product
implicit curve (eqn. 1)
[light]
novel invariant function
implicit curve
[bold]
expanded area
$x = x_0 = 18$
$y = y_0 = 200$
$k = x_0 y_0 = 3600$
$-\frac{dy}{dx} = P = 100/9$
*
y
x

**Figure 3g.**

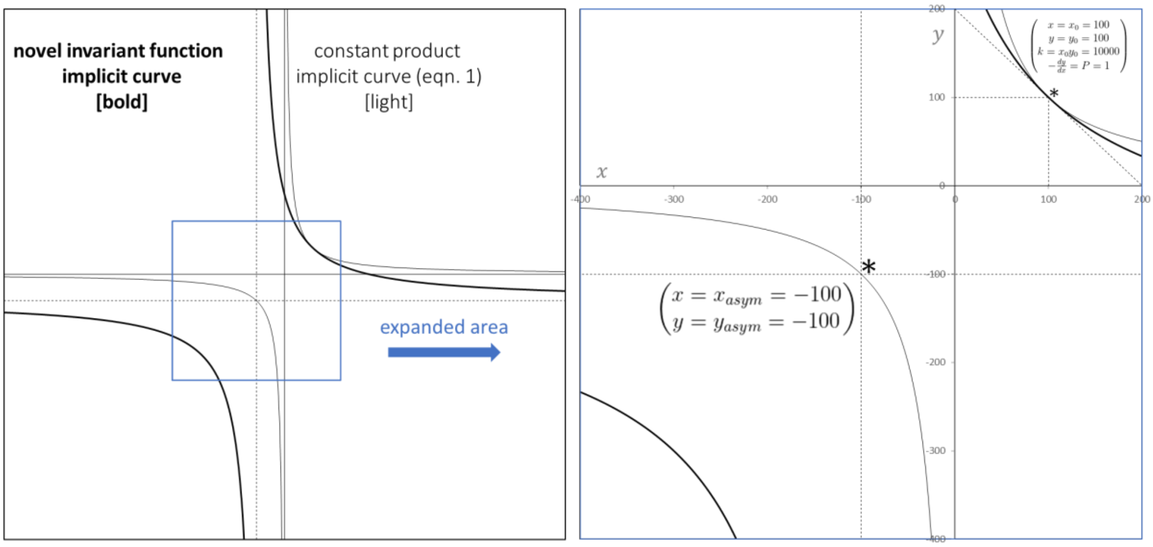



**Figure 4a.**

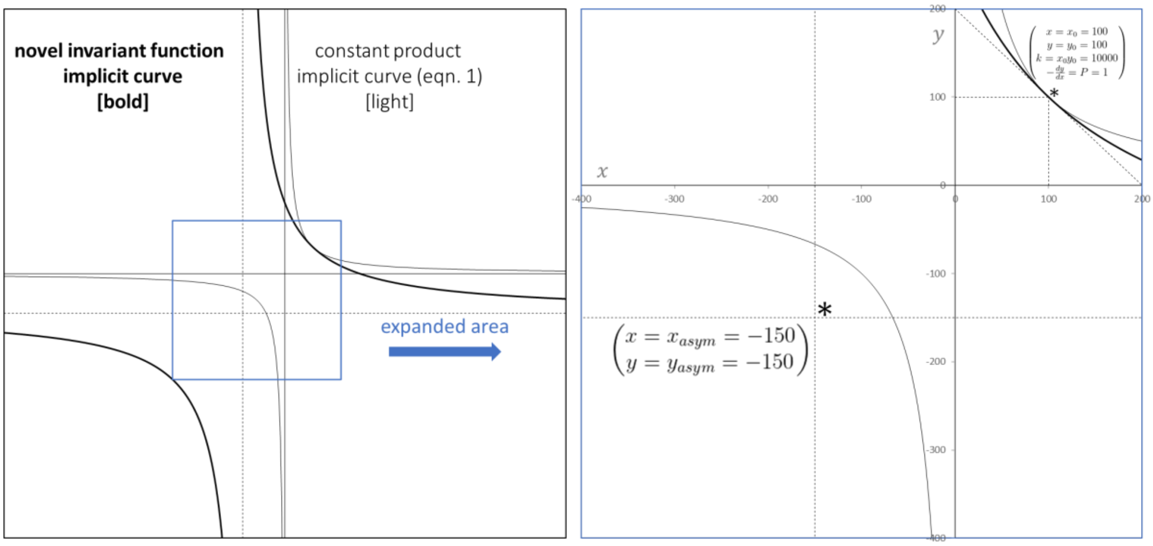



**Figure 4b.**

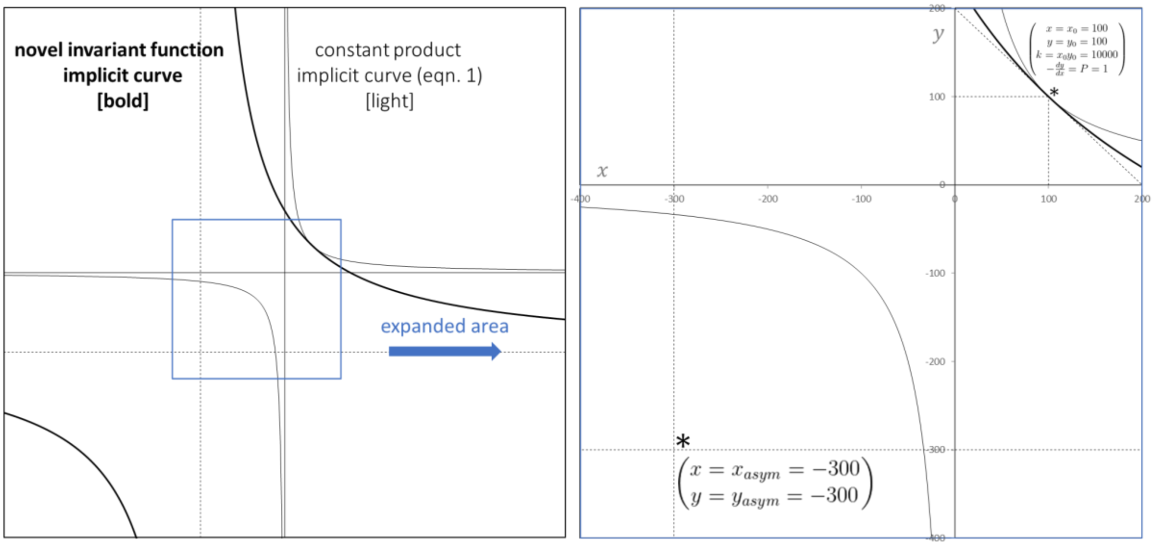



**Figure 4c.**

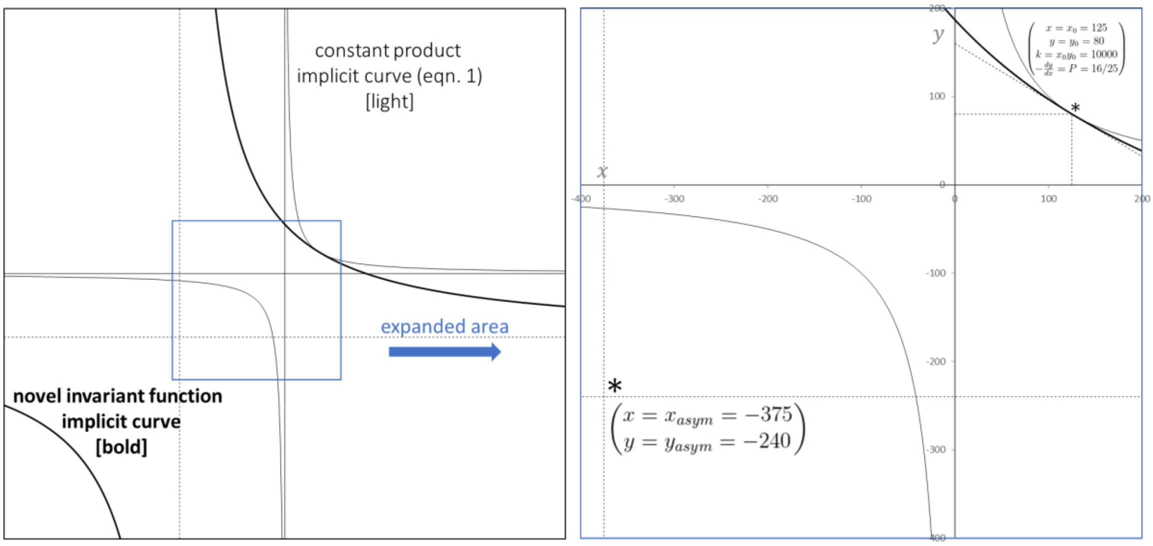



**Figure 4d.**

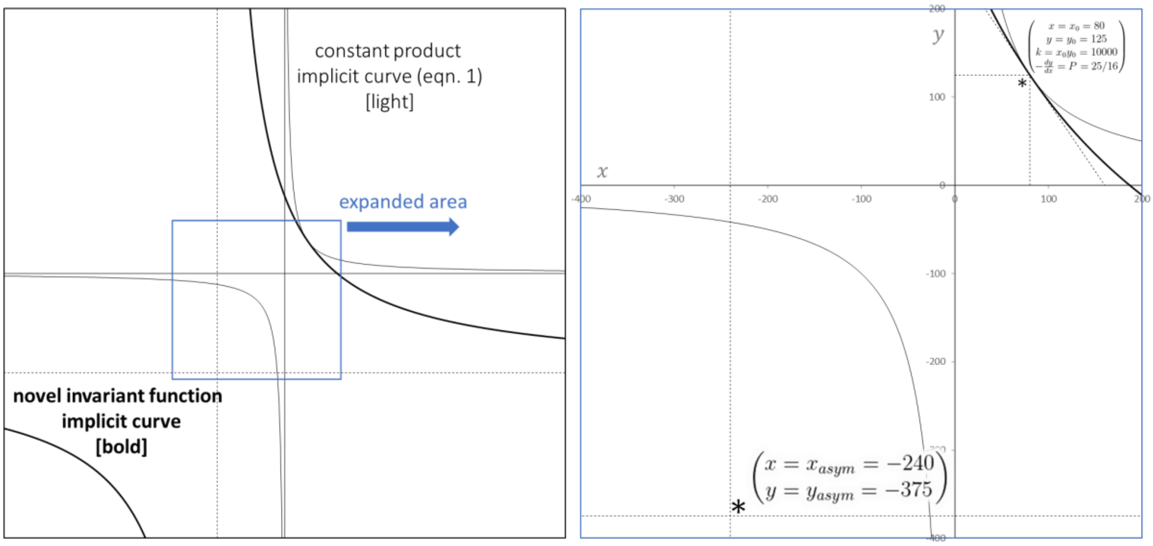



**Figure 4e.**

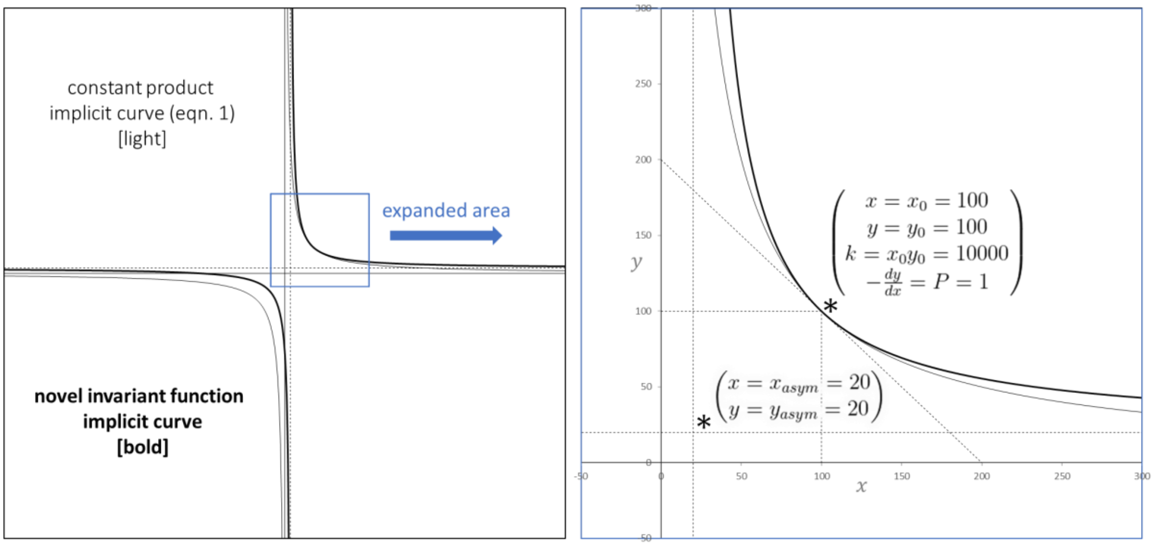



**Figure 4f.**

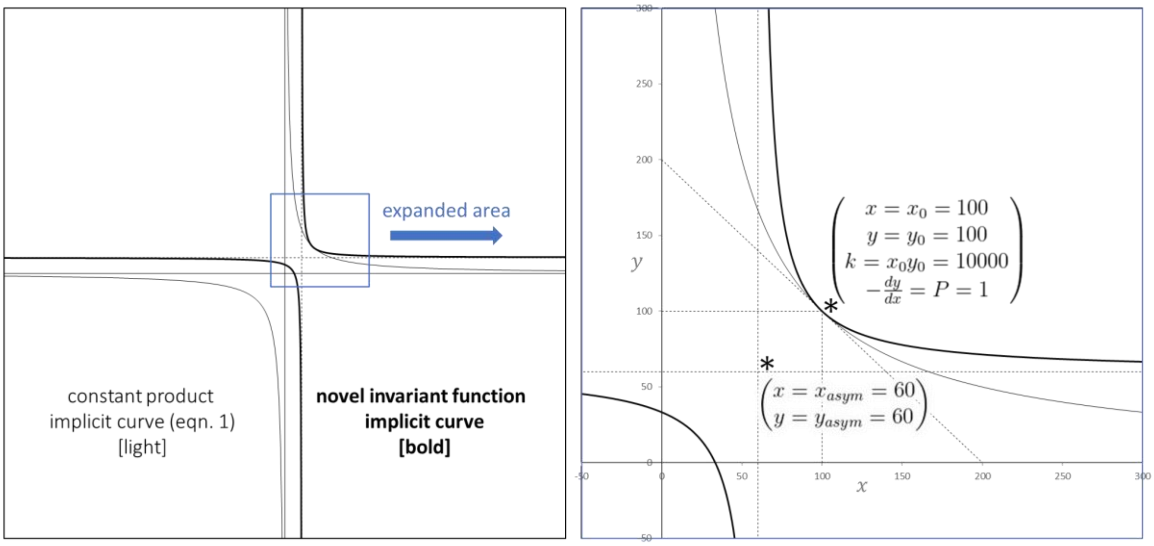



**Figure 4g.**

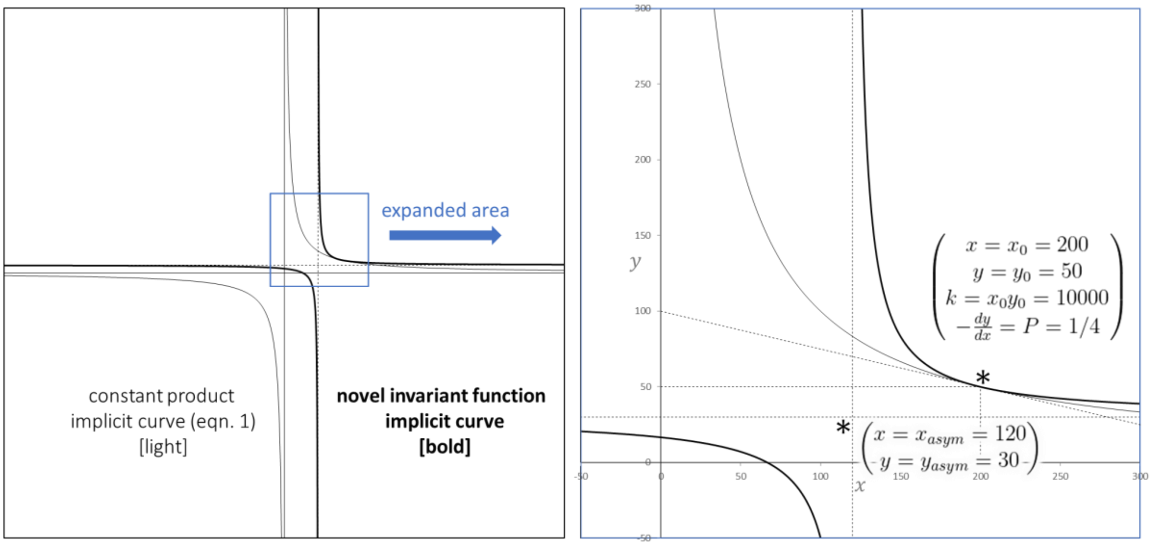



**Figure 4h.**

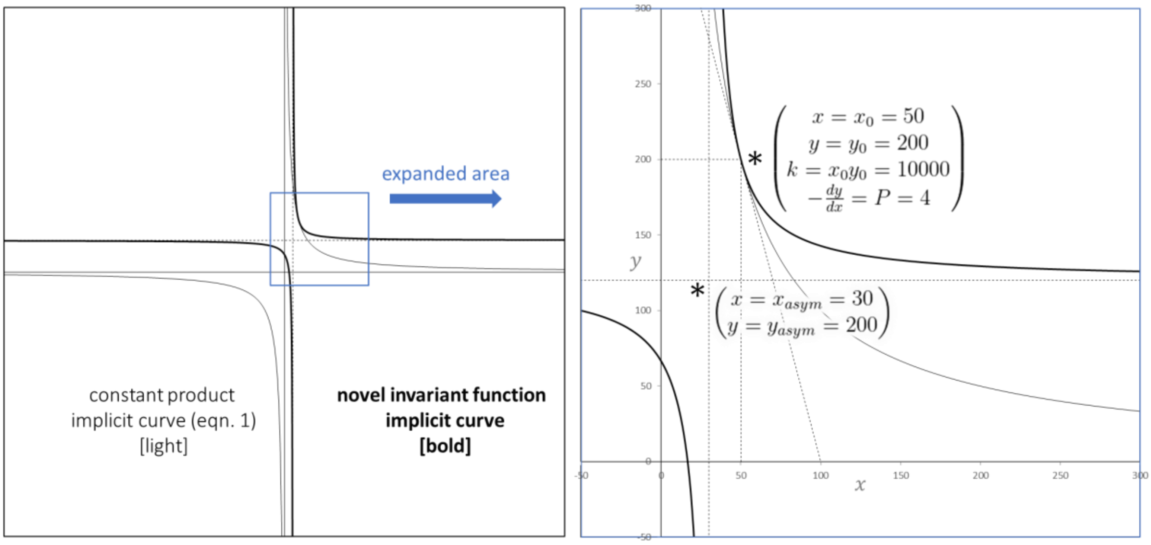



**Figure 4i.**

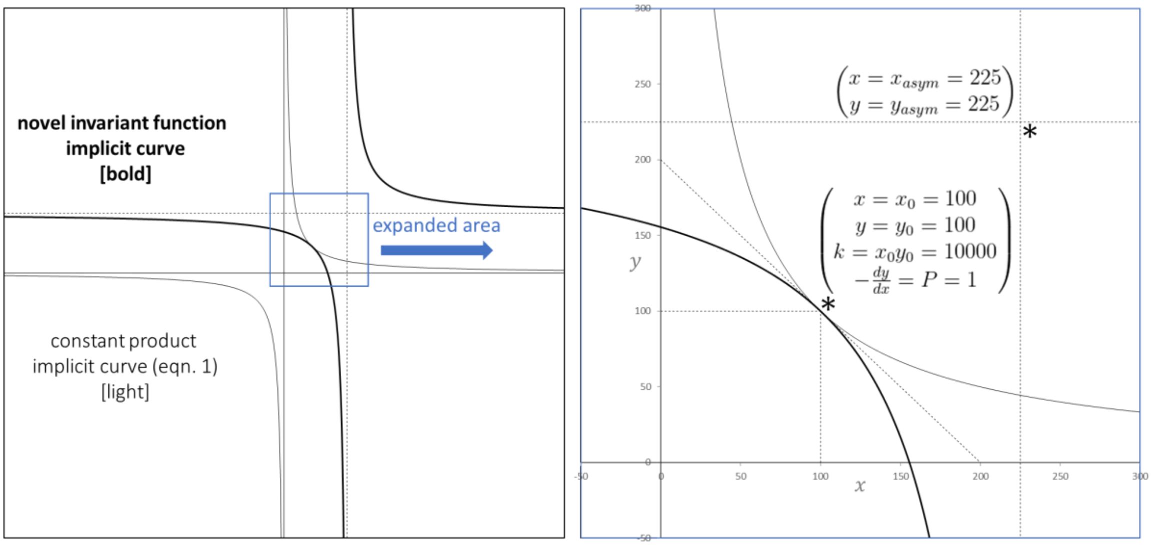



**Figure 4j.**

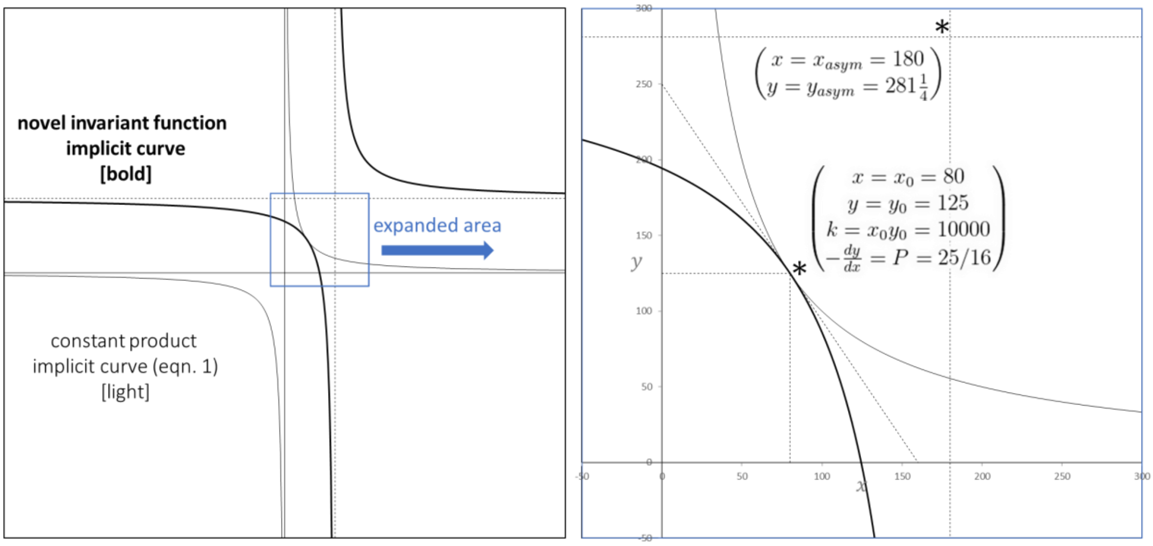



**Figure 4k.**

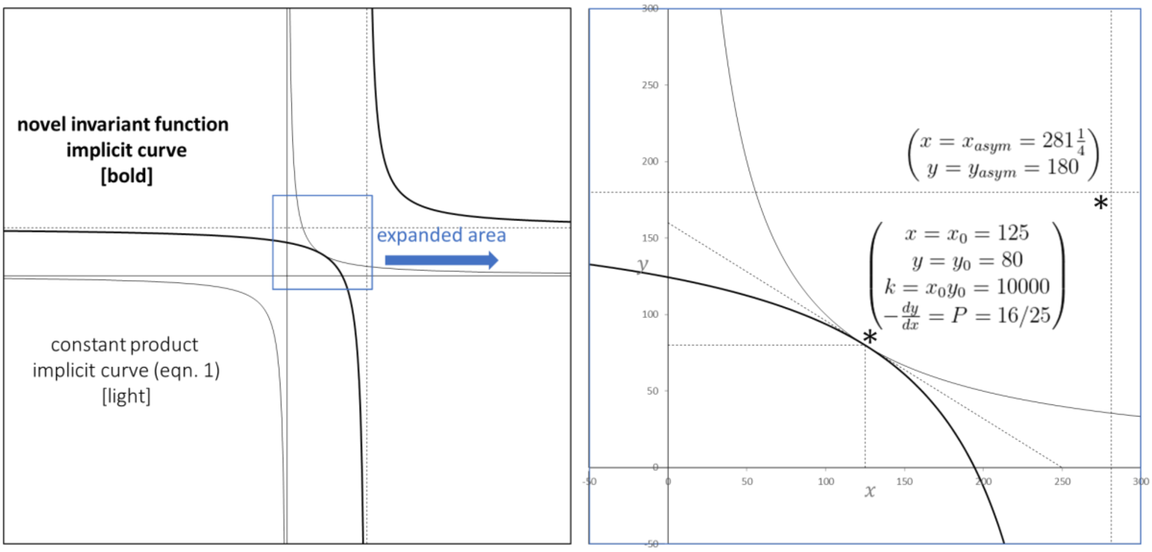



**Figure 4l.**

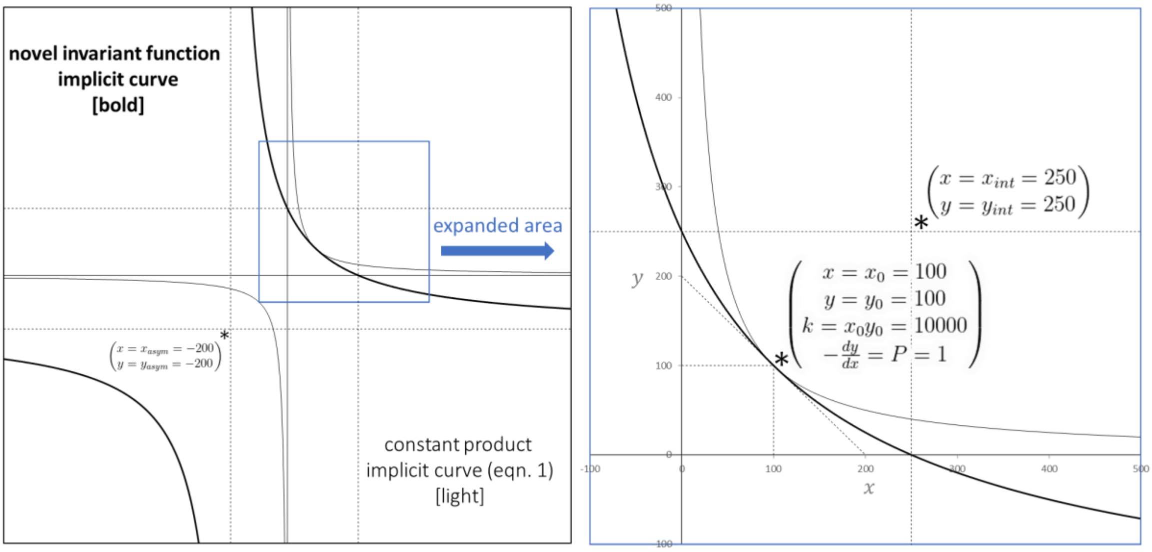



**Figure 5a.**

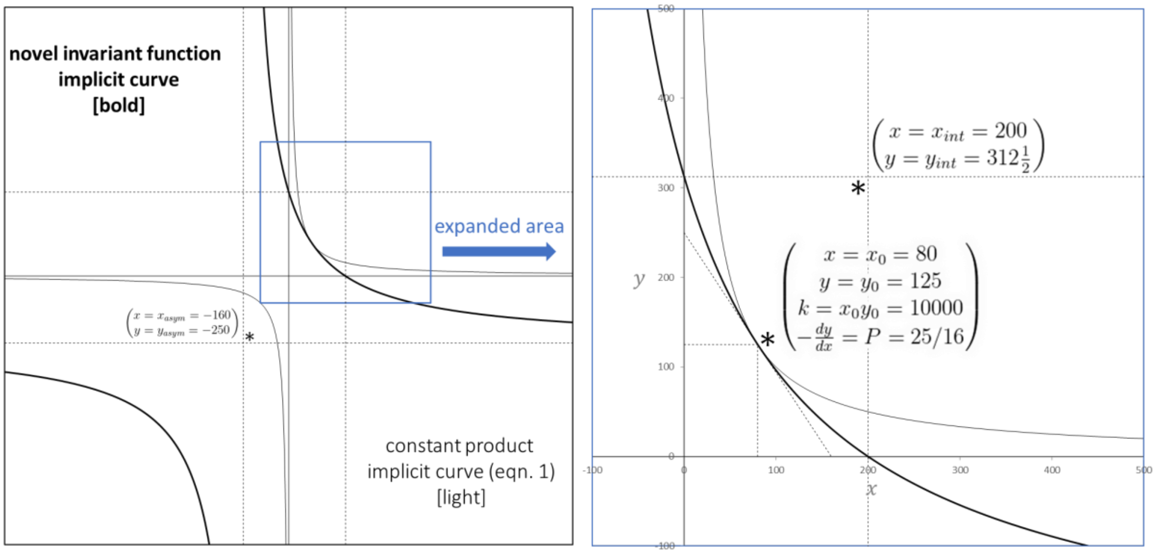



**Figure 5b.**

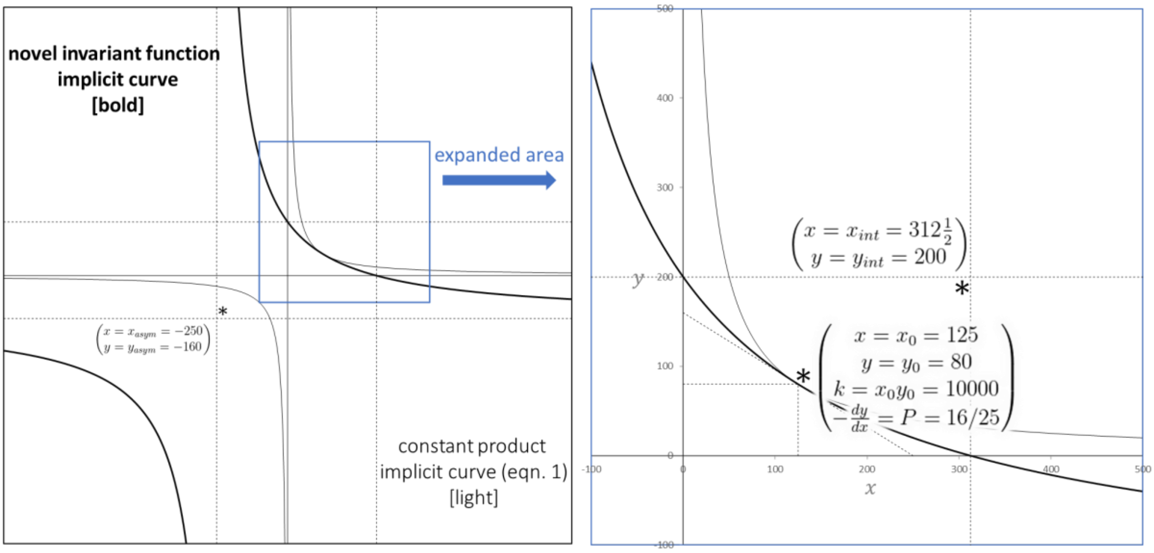



Figure 5c.

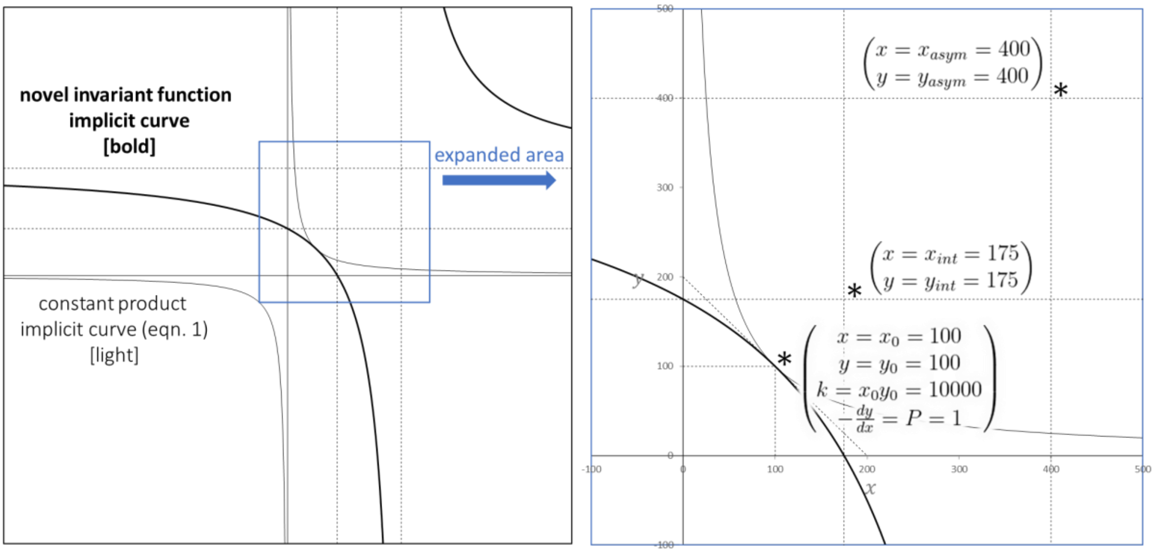



Figure 5d.

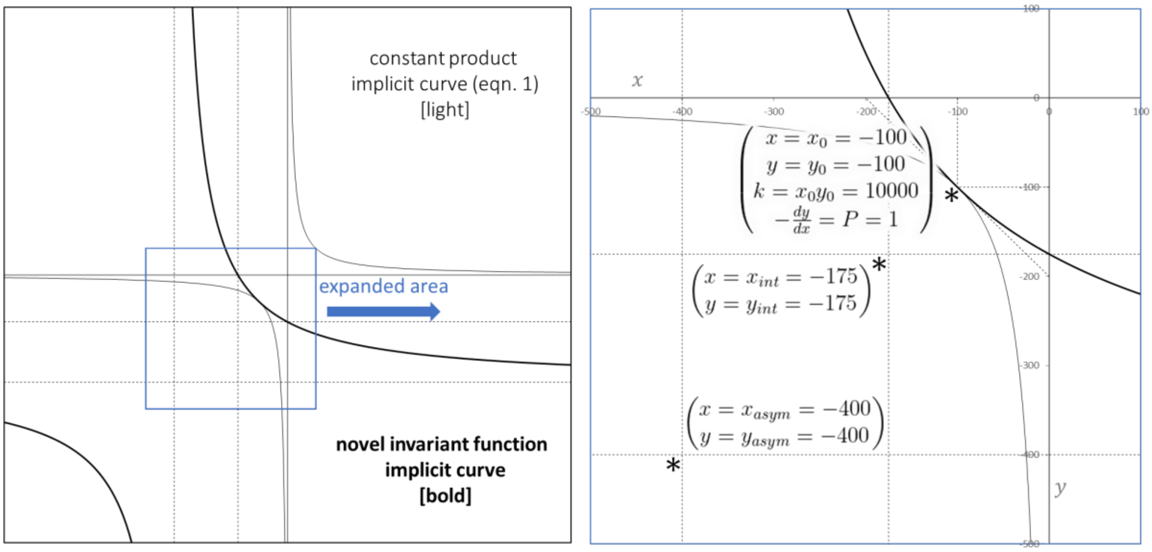



**Figure 5e.**

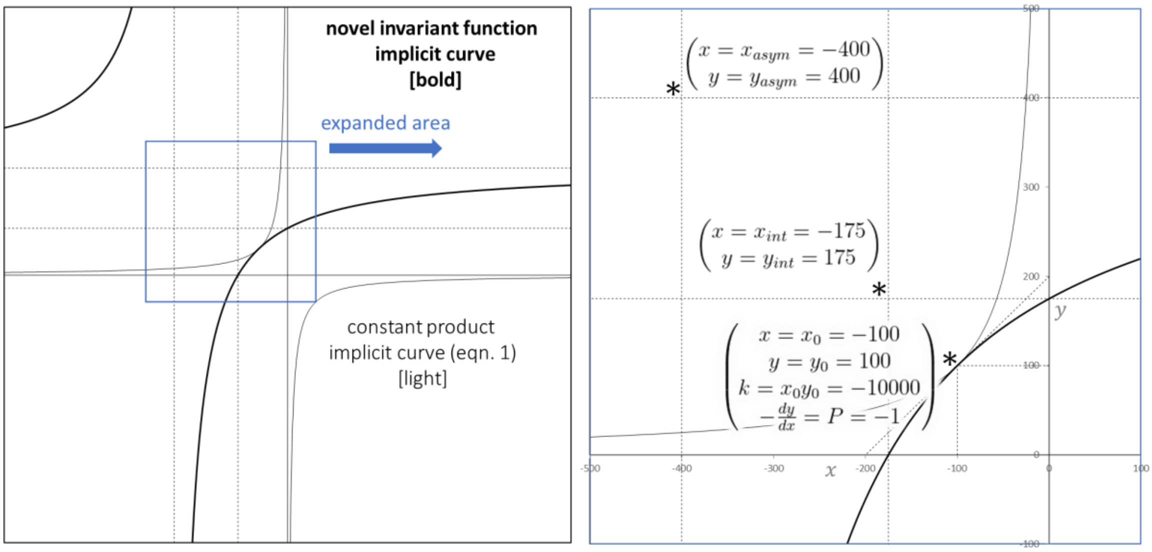
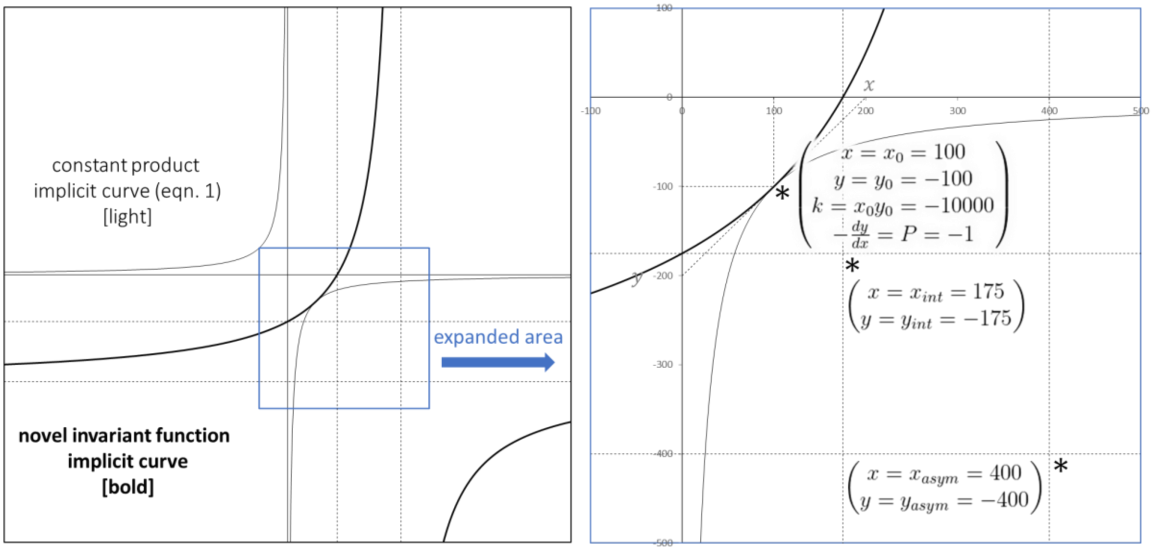



**Figure 5f.**



constant product implicit curve (eqn. 1) [light]
novel invariant function implicit curve [bold]
expanded area
$x$
$y$
-100
0
100
200
300
400
500
-200
-300
-400
-500
*
$x = x_0 = 100$
$y = y_0 = -100$
$k = x_0 y_0 = -10000$
$-\frac{dy}{dx} = P = -1$
*
$x = x_{int} = 175$
$y = y_{int} = -175$
$x = x_{asym} = 400$
$y = y_{asym} = -400$
*


**Figure 5g.**

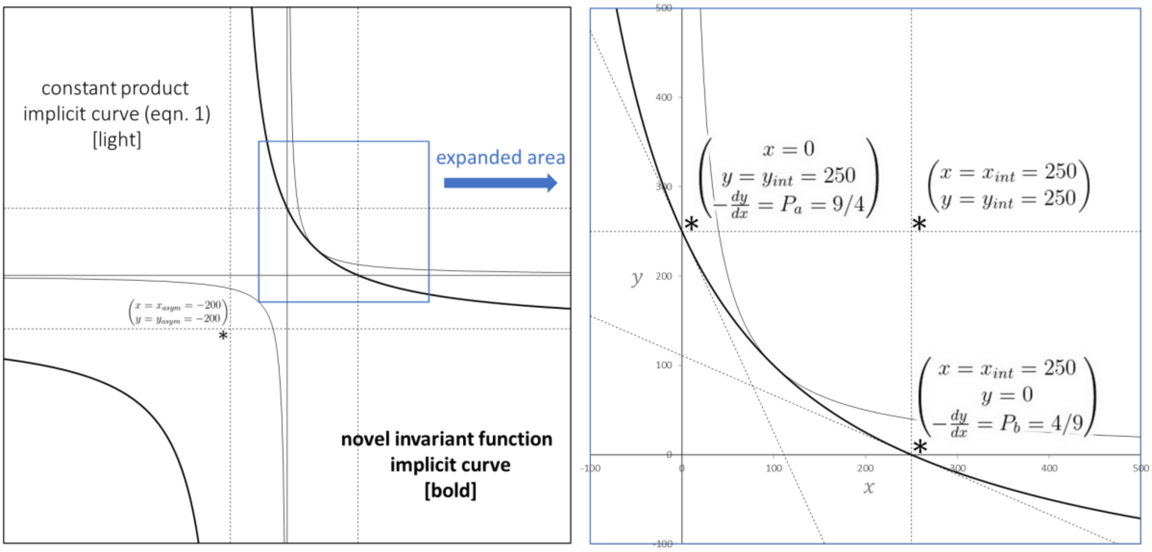



**Figure 6a.**

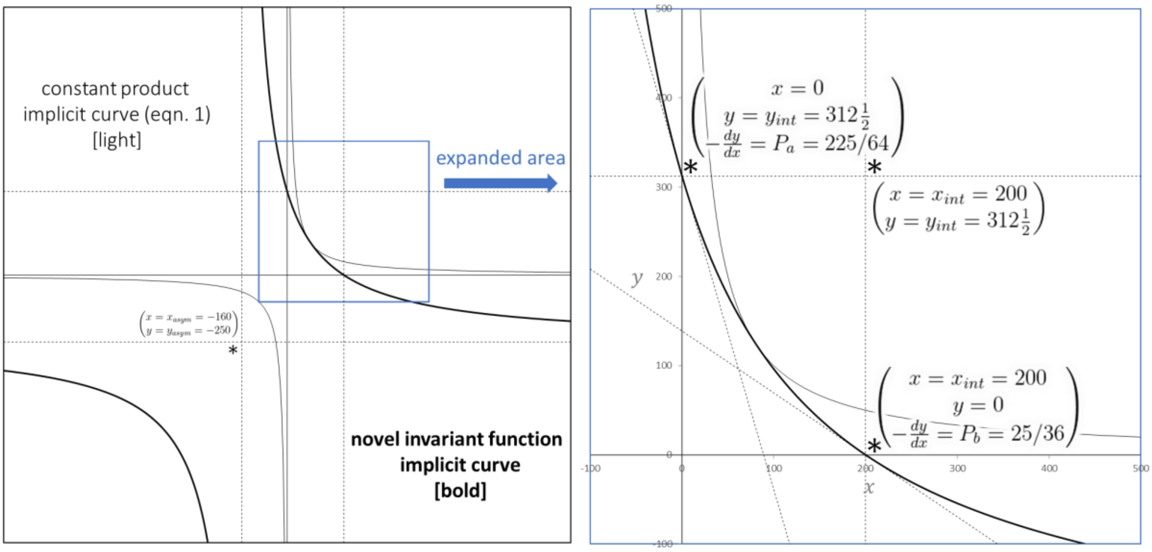



**Figure 6b.**

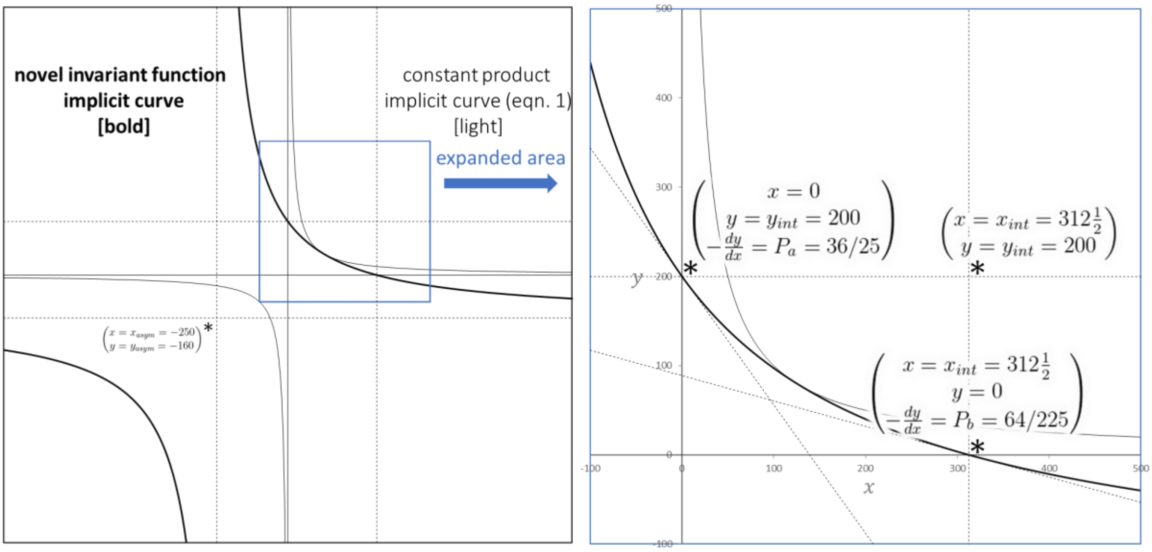



**Figure 6c.**

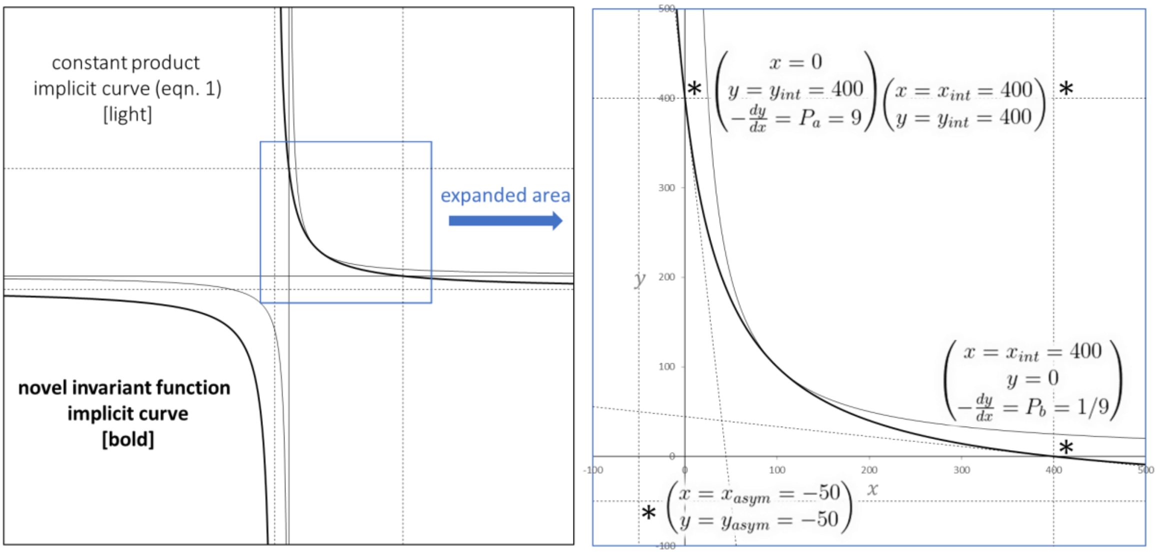



**Figure 6d.**

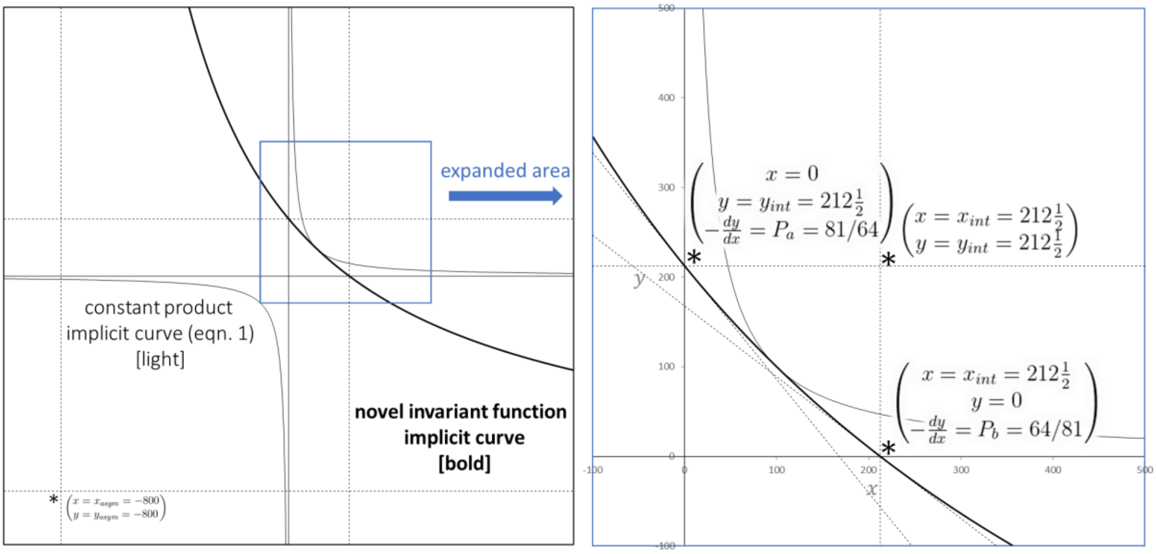



**Figure 6e.**

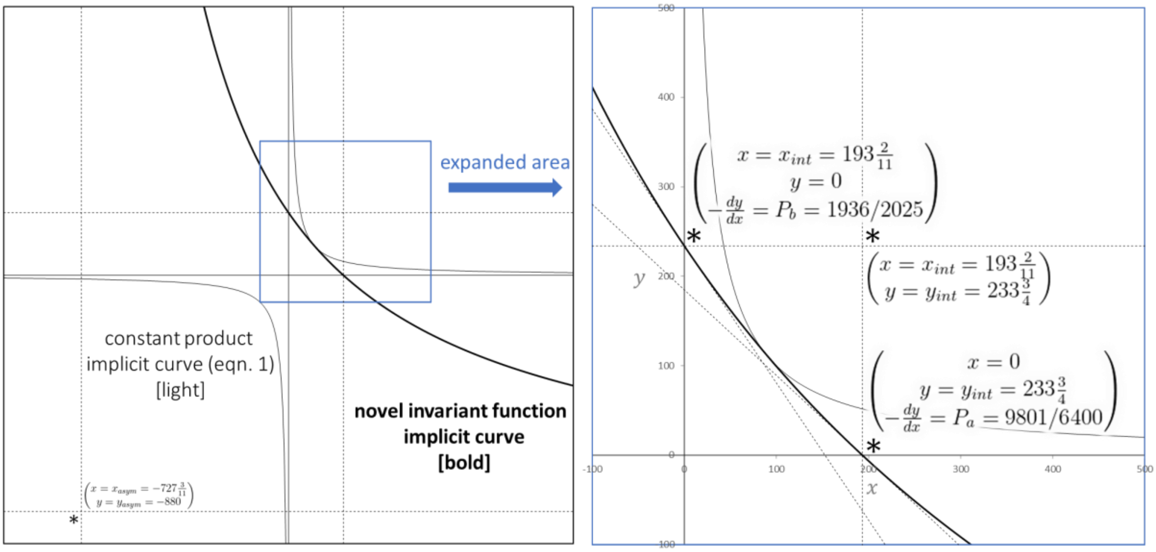



**Figure 6f.**

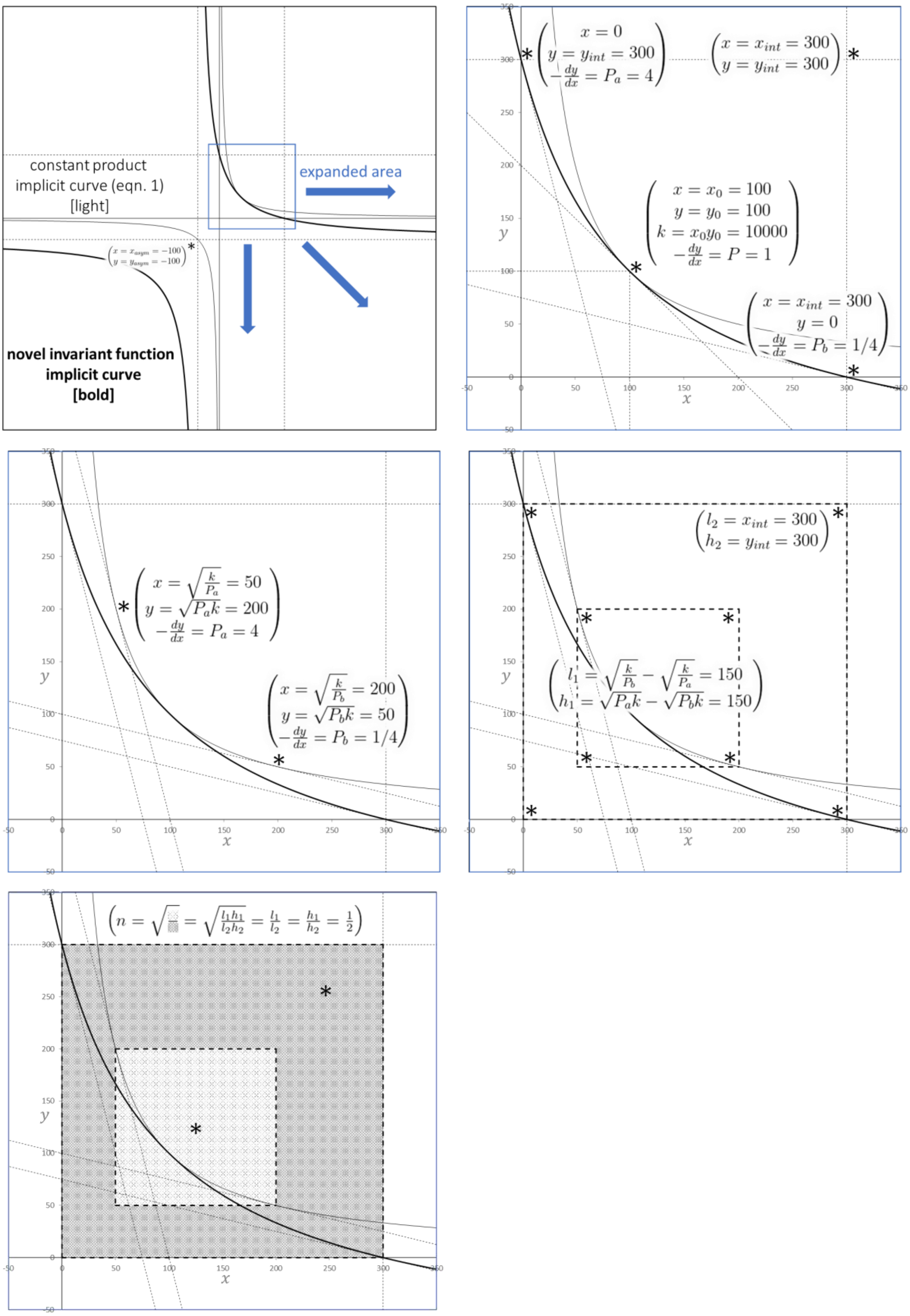
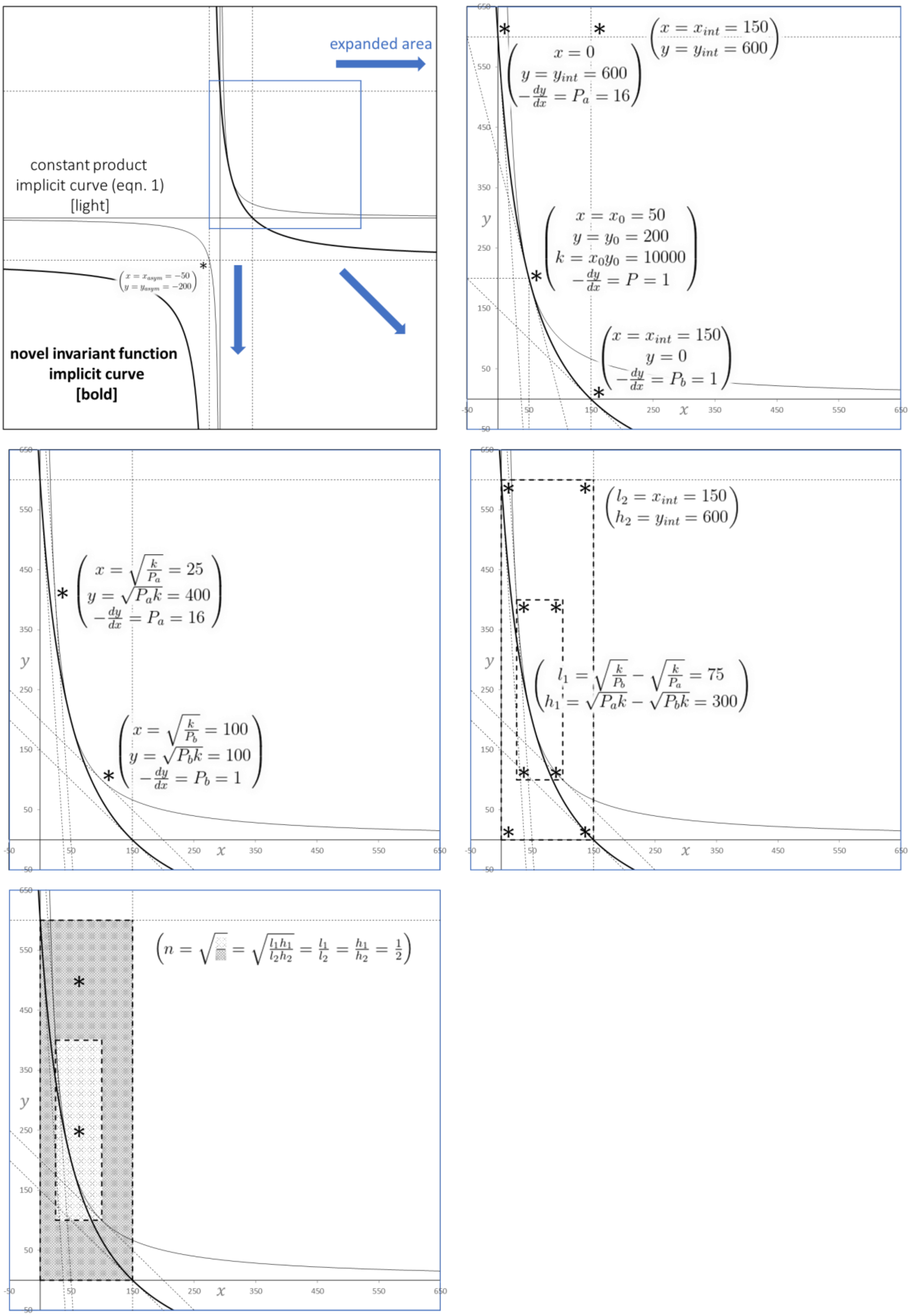

constant product
implicit curve (eqn. 1)
[light]
expanded area
novel invariant function
implicit curve
[bold]
$x = x_{asym} = -100$, $y = y_{asym} = -100$
$x = 0$, $y = y_{int} = 300$, $-\frac{dy}{dx} = P_a = 4$
$x = x_{int} = 300$, $y = y_{int} = 300$
$x = x_0 = 100$, $y = y_0 = 100$, $k = x_0 y_0 = 10000$, $-\frac{dy}{dx} = P = 1$
$x = x_{int} = 300$, $y = 0$, $-\frac{dy}{dx} = P_b = 1/4$
$x = \sqrt{\frac{k}{P_a}} = 50$, $y = \sqrt{P_a k} = 200$, $-\frac{dy}{dx} = P_a = 4$
$x = \sqrt{\frac{k}{P_b}} = 200$, $y = \sqrt{P_b k} = 50$, $-\frac{dy}{dx} = P_b = 1/4$
$l_2 = x_{int} = 300$, $h_2 = y_{int} = 300$
$l_1 = \sqrt{\frac{k}{P_b}} - \sqrt{\frac{k}{P_a}} = 150$, $h_1 = \sqrt{P_a k} - \sqrt{P_b k} = 150$
$n = \sqrt{\frac{l_1 h_1}{l_2 h_2}} = \frac{l_1}{l_2} = \frac{h_1}{h_2} = \frac{1}{2}$
x
y

**Figure 7a.**



expanded area
constant product implicit curve (eqn. 1) [light]
novel invariant function implicit curve [bold]
$x = x_{asym} = -50$, $y = y_{asym} = -200$
$x = x_{int} = 150$, $y = y_{int} = 600$
$x = 0$, $y = y_{int} = 600$, $-\frac{dy}{dx} = P_a = 16$
$x = x_0 = 50$, $y = y_0 = 200$, $k = x_0 y_0 = 10000$, $-\frac{dy}{dx} = P = 1$
$x = x_{int} = 150$, $y = 0$, $-\frac{dy}{dx} = P_b = 1$
$x = \sqrt{\frac{k}{P_a}} = 25$, $y = \sqrt{P_a k} = 400$, $-\frac{dy}{dx} = P_a = 16$
$x = \sqrt{\frac{k}{P_b}} = 100$, $y = \sqrt{P_b k} = 100$, $-\frac{dy}{dx} = P_b = 1$
$l_2 = x_{int} = 150$, $h_2 = y_{int} = 600$
$l_1 = \sqrt{\frac{k}{P_b}} - \sqrt{\frac{k}{P_a}} = 75$, $h_1 = \sqrt{P_a k} - \sqrt{P_b k} = 300$
$n = \sqrt{\frac{l_1 h_1}{l_2 h_2}} = \frac{l_1}{l_2} = \frac{h_1}{h_2} = \frac{1}{2}$
x
y


**Figure 7b.**

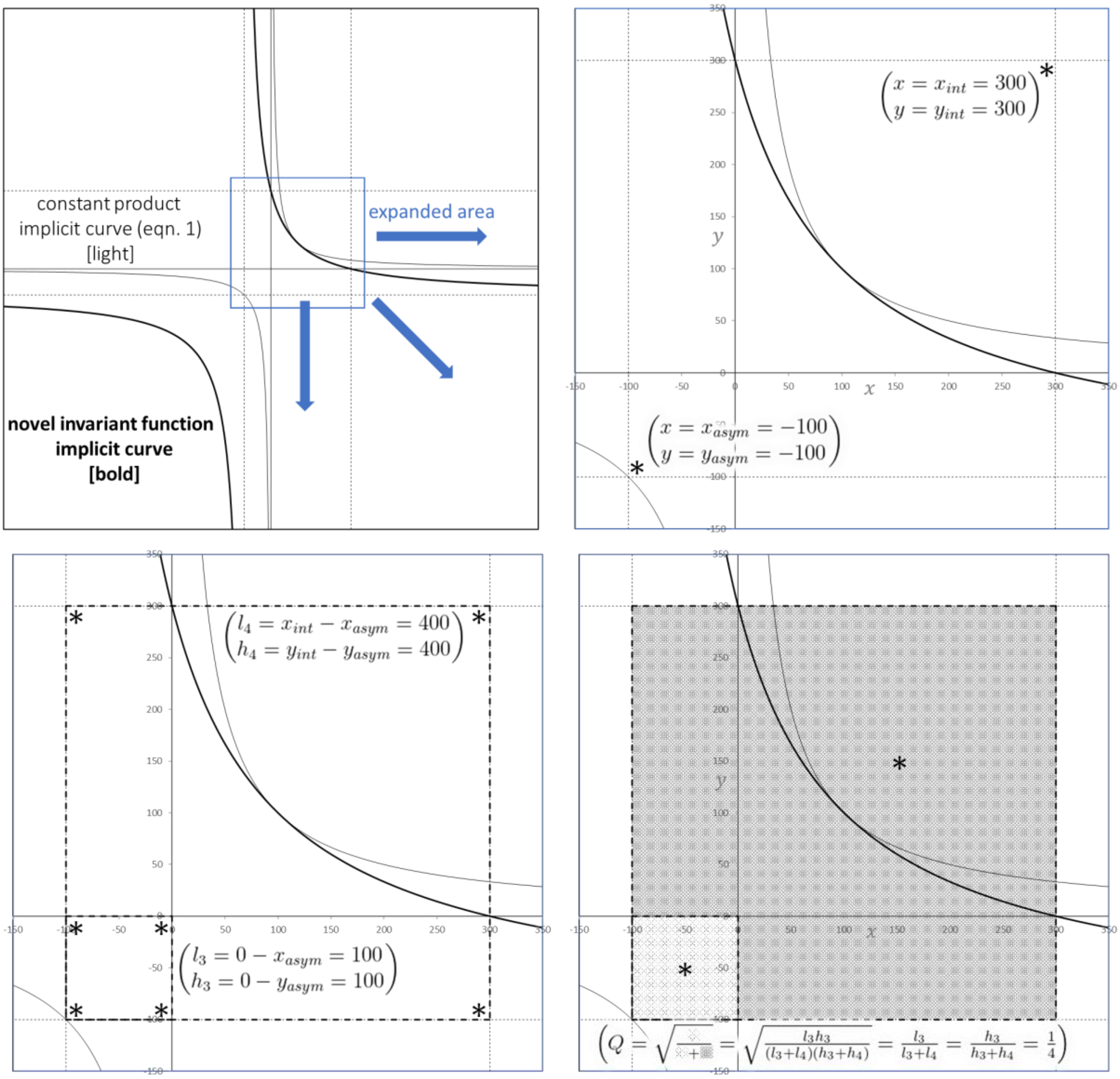

constant product
implicit curve (eqn. 1)
[light]
expanded area
novel invariant function
implicit curve
[bold]
$\begin{pmatrix} x = x_{int} = 300 \\ y = y_{int} = 300 \end{pmatrix}$
$\begin{pmatrix} x = x_{asym} = -100 \\ y = y_{asym} = -100 \end{pmatrix}$
$\begin{pmatrix} l_4 = x_{int} - x_{asym} = 400 \\ h_4 = y_{int} - y_{asym} = 400 \end{pmatrix}$
$\begin{pmatrix} l_3 = 0 - x_{asym} = 100 \\ h_3 = 0 - y_{asym} = 100 \end{pmatrix}$
$\left(Q = \sqrt{\frac{l_3 h_3}{(l_3+l_4)(h_3+h_4)}} = \frac{l_3}{l_3+l_4} = \frac{h_3}{h_3+h_4} = \frac{1}{4}\right)$

**Figure 7c.**

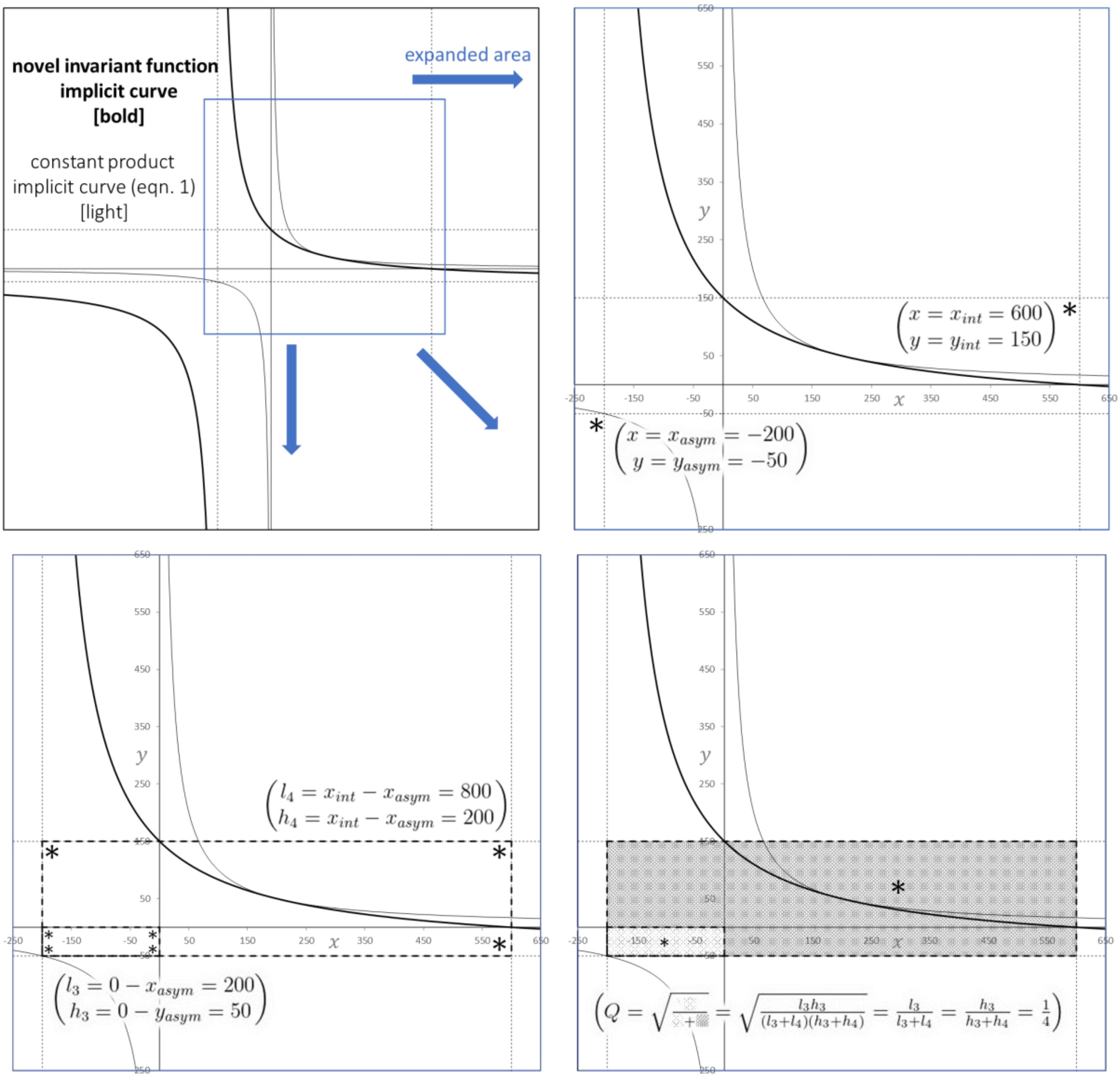

novel invariant function
implicit curve
[bold]
constant product
implicit curve (eqn. 1)
[light]
expanded area
$\begin{pmatrix} x = x_{int} = 600 \\ y = y_{int} = 150 \end{pmatrix}$
$\begin{pmatrix} x = x_{asym} = -200 \\ y = y_{asym} = -50 \end{pmatrix}$
$\begin{pmatrix} l_4 = x_{int} - x_{asym} = 800 \\ h_4 = x_{int} - x_{asym} = 200 \end{pmatrix}$
$\begin{pmatrix} l_3 = 0 - x_{asym} = 200 \\ h_3 = 0 - y_{asym} = 50 \end{pmatrix}$
$\left(Q = \sqrt{\frac{l_3 h_3}{(l_3+l_4)(h_3+h_4)}} = \frac{l_3}{l_3+l_4} = \frac{h_3}{h_3+h_4} = \frac{1}{4}\right)$

**Figure 7d.**

A vast expanse of additional rearrangements of the equations presented above can be generated from the geometric identities contained within the scope of our invention, and their implicit curves comprise a specific set of possible exchange algorithms. For all positive values of $x$ and $y$, which may represent the balances of tokens in a smart contract, or any similar arrangement on a distributed ledger, the application of the novel invariant function provides a map of exchange rates that encompasses a novel DEX technology.

The overparameterization described herein enables the creation of subsets of new parameters that reference those defined in the sections above, from which specific forms of the novel invariant function can be explicitly derived. This concept is exemplified in the following sections with reference to specific examples, which represent a subset of a much larger collection of possible groupings of curve parameters and associated rearrangements of the general formula. Such flexibility allows for application-specific considerations to be addressed, including but not necessarily limited to calculation accuracy in the absence of floating-point arithmetic, overflow and underflow, unsigned integers, computational complexity, storage and memory requirements, or any limitation imposed either directly or indirectly by the hardware or software on which the outputs of the function are determined. This aspect to the invention is critically important, and is demonstrated with specific examples below.

The general theory can be understood heuristically. One set of implicit curves encompassed by the present invention is comprised of those which can be selected with reference to a coordinate of the implicit curve of **eqns. 1a-c**, and some scaling factor. While the parameters $n$, $u$ and $Q$ are the natural expression of the scaling factor, it is possible to construct forms of the invariant function that feature none of these. For example, the invariant functions described by **eqn. 15, 16a-c,** and **17a-c** are perfectly serviceable despite their lack of direct reference to $n$, $u$ or $Q$, as demonstrated by their elaboration into the marginal exchange formulas (**eqns. 16d-e** and **17d-e**), their swap formulas (**eqns. 16f-g** and **17f-g**), and gross exchange rate formulas (**eqns. 16h-i** and **17h-i**). Such a construction is possible because the three parameters ($x_0$, $y_0$, and either $y_{\text{int}}$ or $x_{\text{int}}$) each contribute partial positional *and* scaling information with respect to each other.

$$\frac{(x_{int} - x_0)}{x_0} = \frac{(y_{int} - y_0)}{y_0} = \frac{(x - x_0)(y_0 - y)}{x_0 y_0 - xy} \quad \textbf{eqn. 15}$$

$$(y_{int} - y_0)(x_0 y_0 - xy) = y_0 (x - x_0)(y_0 - y) \quad \textbf{eqn. 16a}$$

$$y = \frac{y_0 (x_0 y_{int} - x y_0)}{x_0 y_0 - x (2y_0 - y_{int})} \quad \textbf{eqn. 16b}$$

$$x = \frac{x_0 y_0 (y_{int} - y)}{y_0^2 - y (2y_0 - y_{int})} \quad \textbf{eqn. 16c}$$

$$-\frac{dy}{dx} = \frac{x_0 y_0 (y_0^2 + y_{int} (y_{int} - 2y_0))}{(x (y_{int} - 2y_0) + x_0 y_0)^2} \quad \textbf{eqn. 16d}$$

$$-\frac{dy}{dx} = \frac{(y (y_{int} - 2y_0) + y_0^2)^2}{x_0 y_0 (y_{int} - y_0)^2} \quad \textbf{eqn. 16e}$$

$$-\Delta y = \frac{\Delta x x_0 y_0 (y_{int} - y_0)^2}{(x y_{int} + y_0 (x_0 - 2x))(x y_{int} + y_0 (x_0 - 2x) - \Delta x (2y_0 - y_{int}))} \quad \textbf{eqn. 16f}$$

$$-\Delta y = \frac{\Delta x\left(y\left(y_{int}-2y_0\right)+y_0^2\right)^2}{\Delta x\left(y_{int}-2y_0\right)\left(y\left(y_{int}-2y_0\right)-y_0^2\right)+x_0y_0\left(y_{int}-y_0\right)^2} \quad \textbf{eqn. 16g}$$

$$-\frac{\Delta y}{\Delta x} = \frac{x_0y_0\left(y_{int}-y_0\right)^2}{\left(xy_{int}+y_0\left(x_0-2x\right)\right)\left(xy_{int}+y_0\left(x_0-2x\right)-\Delta x\left(2y_0-y_{int}\right)\right)} \quad \textbf{eqn. 16h}$$

$$-\frac{\Delta y}{\Delta x} = \frac{\left(y\left(y_{int}-2y_0\right)+y_0^2\right)^2}{\Delta x\left(y_{int}-2y_0\right)\left(y\left(y_{int}-2y_0\right)-y_0^2\right)+x_0y_0\left(y_{int}-y_0\right)^2} \quad \textbf{eqn. 16i}$$

$$\left(x_{int}-x_0\right)\left(x_0y_0-xy\right) = x_0\left(x-x_0\right)\left(y_0-y\right) \quad \textbf{eqn. 17a}$$

$$y = \frac{x_0y_0\left(x_{int}-x\right)}{x\left(x_{int}-2x_0\right)+x_0^2} \quad \textbf{eqn. 17b}$$

$$x = \frac{x_0\left(x_{int}y_0-x_0y\right)}{x_0y_0+y\left(x_{int}-2x_0\right)} \quad \textbf{eqn. 17c}$$

$$-\frac{dy}{dx} = \frac{x_0y_0\left(x_{int}-x_0\right)^2}{\left(x\left(x_{int}-2x_0\right)+x_0^2\right)^2} \quad \textbf{eqn. 17d}$$

$$-\frac{dy}{dx} = \frac{\left(x_0y_0+y\left(x_{int}-2x_0\right)\right)^2}{x_0y_0\left(x_{int}-x_0\right)^2} \quad \textbf{eqn. 17e}$$

$$-\Delta y = \frac{\Delta x x_0y_0\left(x_{int}-x_0\right)^2}{\left(x_0^2+\left(\Delta x+x\right)\left(x_{int}-2x_0\right)\right)\left(x\left(x_{int}-2x_0\right)+x_0^2\right)} \quad \textbf{eqn. 17f}$$

$$-\Delta y = \frac{\Delta x\left(x_0y_0+y\left(x_{int}-2x_0\right)\right)^2}{x_oy_0\left(x_{int}-x_0\right)^2-\Delta x\left(x_{int}-2x_0\right)\left(y\left(x_{int}-2x_0\right)-x_0y_0\right)} \quad \textbf{eqn. 17g}$$

$$-\frac{\Delta y}{\Delta x} = \frac{x_0y_0\left(x_{int}-x_0\right)^2}{\left(x_0^2+\left(\Delta x+x\right)\left(x_{int}-2x_0\right)\right)\left(x\left(x_{int}-2x_0\right)+x_0^2\right)} \quad \textbf{eqn. 17h}$$

$$-\frac{\Delta y}{\Delta x} = \frac{\left(x_0y_0+y\left(x_{int}-2x_0\right)\right)^2}{x_0y_0\left(x_{int}-x_0\right)-\Delta x\left(x_{int}-2x_0\right)\left(y\left(x_{int}-2x_0\right)-x_0y_0\right)} \quad \textbf{eqn. 17i}$$

Therefore, it is possible to define new fundamental scaling constants from the algebraic and geometric identities disclosed here for the novel invariant function. This is an important advance from a technological perspective, as it allows for discrete forms of the general theory to created to suit specific implementation constraints. For the two cases considered for the set resulting from the elaboration of **eqn. 15**, it is trivial to define the new scaling term, *R*, and demonstrate its congruence with the general theory (**eqns. 18a-b**). The explicit forms of this simplification are presented in **eqns. 19a-i**.

Invariant functions of this kind can be easily demonstrated as belonging to the same superset and are contained within the scope of the invention disclosed here, which is itself a technological innovation upon our prior art (**Appendix 1**).

$$R = \frac{(x_{int} - x_0)}{x_0} = \frac{(y_{int} - y_0)}{y_0}$$ **eqn. 18a**

$$\frac{(1-n)^2}{n^2(2-n)^2} = \frac{P}{P_a + P_b - 2P} = \frac{Q}{(1-Q)^2} = \frac{R^2}{(R-1)^2(R+1)^2}$$ **eqn. 18b**

$$R(x_0y_0 - xy) = (x - x_0)(y_0 - y)$$ **eqn. 19a**

$$y = \frac{y_0(x_0(R+1) - x)}{x(R-1) + x_0}$$ **eqn. 19b**

$$x = \frac{x_0(y_0(R+1) - y)}{y(R-1) + x_0}$$ **eqn. 19c**

$$-\frac{dy}{dx} = \frac{R^2x_0y_0}{(x(R-1) + x_0)^2}$$ **eqn. 19d**

$$-\frac{dy}{dx} = \frac{x_0y_0(y(R-1) + y_0^2)^2}{R^2}$$ **eqn. 19e**

$$-\Delta y = \frac{\Delta x R^2 x_0 y_0}{(x_0 + (\Delta x + x)(R-1))(x(R-1) + x_0)}$$ **eqn. 19f**

$$-\Delta y = \frac{\Delta x(y(R-1) + y_0)^2}{\Delta x(R-1)(y(R-1) + y_0) + R^2x_0y_0}$$ **eqn. 19g**

$$-\frac{\Delta y}{\Delta x} = \frac{R^2x_0y_0}{(x_0 + (\Delta x + x)(R-1))(x(R-1) + x_0)}$$ **eqn. 19h**

$$-\frac{\Delta y}{\Delta x} = \frac{(y(R-1) + y_0)^2}{\Delta x(R-1)(y(R-1) + y_0) + R^2x_0y_0}$$ **eqn. 19i**

Trivial combinations of the parameters described herein are also within the scope of the invention. In one embodiment, let the parameter *S* be defined as the difference between the square roots of the gradients at the *y*- and *x*-intercepts, which through algebraic manipulation is shown to have equivalence with other fundamental parameters *P*, *Q*, and *n* (**eqn. 20a-b**). Then, redefinition of $P_b$ as $B^2$ (**eqns. 20c**) allows for the construction of a new equation set (**eqns. 21a-i**).

$$S = \sqrt{P_a} - \sqrt{P_b} = \sqrt{\frac{P}{Q}}(1 - Q) = \frac{\sqrt{P}n(2-n)}{1-n}$$ **eqn. 20a**

$$\frac{(1-n)^2}{n^2(2-n)^2} = \frac{P}{P_a + P_b - 2P} = \frac{Q}{(1-Q)^2} = \frac{R^2}{(R-1)^2(R+1)^2} = \frac{P}{S^2}$$ **eqn. 20b**

$$B^2 = P_b$$ **eqn. 20c**

$$y\left(x\left(BR+R^2\right)+y_{int}\right)=y_{int}\left(y_{int}-x\left(BR+B^2\right)\right) \quad \textbf{eqn. 21a}$$

$$y=\frac{y_{int}\left(y_{int}-Bx\left(B+S\right)\right)}{x\left(BS+S^2\right)+y_{int}} \quad \textbf{eqn. 21b}$$

$$x=\frac{y_{int}\left(y_{int}-y\right)}{\left(B+S\right)\left(By_{int}+Sy\right)} \quad \textbf{eqn. 21c}$$

$$-\frac{dy}{dx}=\frac{y_{int}^2\left(B+S\right)^2}{\left(x\left(BS+S^2\right)+y_{int}\right)^2} \quad \textbf{eqn. 21d}$$

$$-\frac{dy}{dx}=\frac{\left(By_{int}+Sy\right)^2}{y_{int}^2} \quad \textbf{eqn. 21e}$$

$$-\Delta y=\frac{\Delta x y_{int}^2\left(B+S\right)^2}{\left(x\left(BS+S^2\right)+y_{int}\right)\left(S\left(B+S\right)\left(\Delta x+x\right)+y_{int}\right)} \quad \textbf{eqn. 21f}$$

$$-\Delta y=\frac{\Delta x\left(By_{int}+Sy\right)^2}{S\Delta x\left(By_{int}+Sy\right)+y_{int}^2} \quad \textbf{eqn. 21g}$$

$$-\frac{\Delta y}{\Delta x}=\frac{y_{int}^2\left(B+S\right)^2}{\left(x\left(BS+S^2\right)+y_{int}\right)\left(S\left(B+S\right)\left(\Delta x+x\right)+y_{int}\right)} \quad \textbf{eqn. 21h}$$

$$-\frac{\Delta y}{\Delta x}=\frac{y_{int}^2\left(By_{int}+Sy\right)^2}{\Delta x\left(BSy_{int}+S^2y\right)+y_{int}^2} \quad \textbf{eqn. 21i}$$

This set is included to emphasize the brevity and computational tractability of its forms, especially its swap formula, **eqn. 21g**, and its marginal price formula **eqn. 21e**, which showcases the advantages afforded by our overparameterization and specific algebraic, and geometric elaboration of the general theory. This set is adopted to illustrate the second part of this invention, *asymmetric liquidity pools*, at the conclusion of this section.

The examples provided hitherto are for a discrete case where fundamental parameters of the invariant function are assumed to be fixed values, which is not a requirement Any function parameter may also be functions of internal variables including but not necessarily limited to the *x*- or *y*-coordinates, or any combination of the *x*- and *y*- coordinates and other parameters, or functions of external variables including but not necessarily limited to data delivered by another smart contract, application programmable interface (API), or a system or network of oracles.

To exemplify the scope of the invention, **eqn. 22** substitutes *n* for *f*(*x*). Importantly, the novel invariant function maintains a point on its graph with coordinates ($x_0$, $y_0$) coincident with the implicit curve of **eqn. 1**, and the slope of the tangent line at this point is equal in both (**Figure ba-c**).

$$n=cos\left(\frac{\pi x}{200}\right)+1 \quad \textbf{eqn. 22}$$

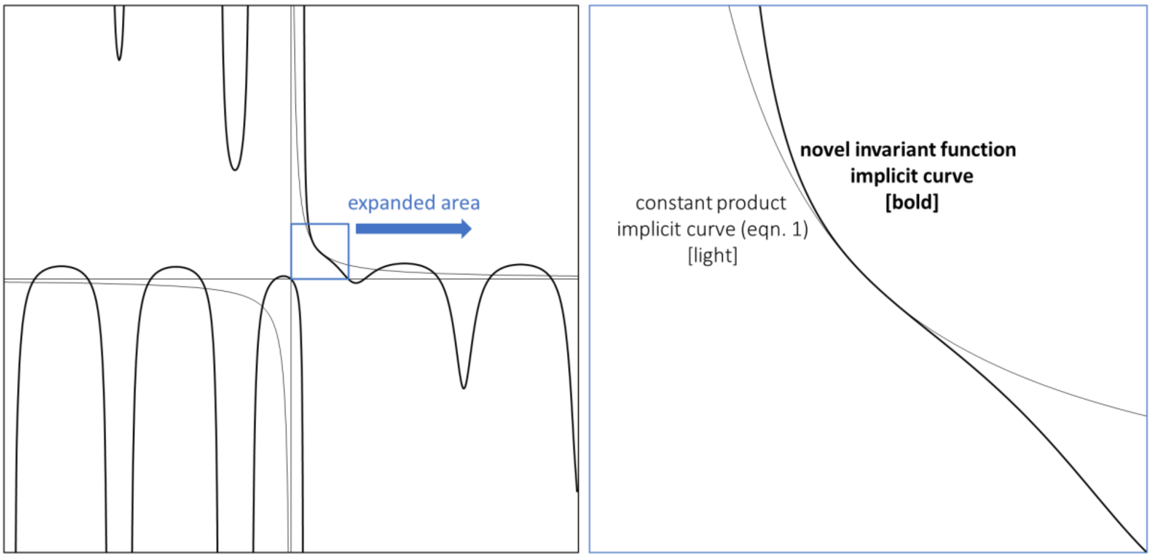



**Figure 8a.**

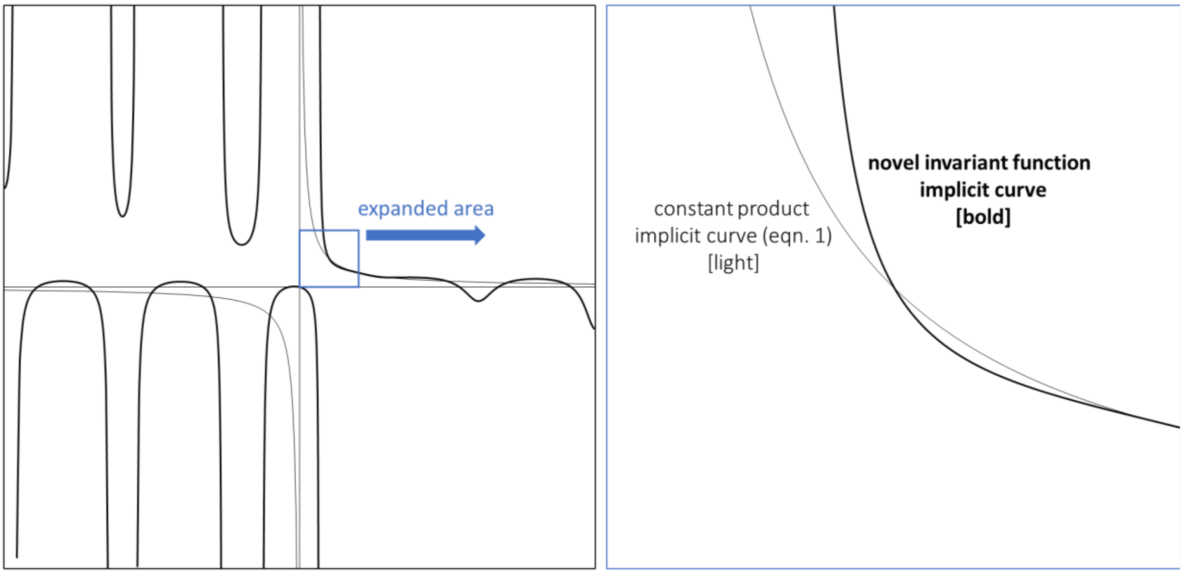



**Figure 8b.**

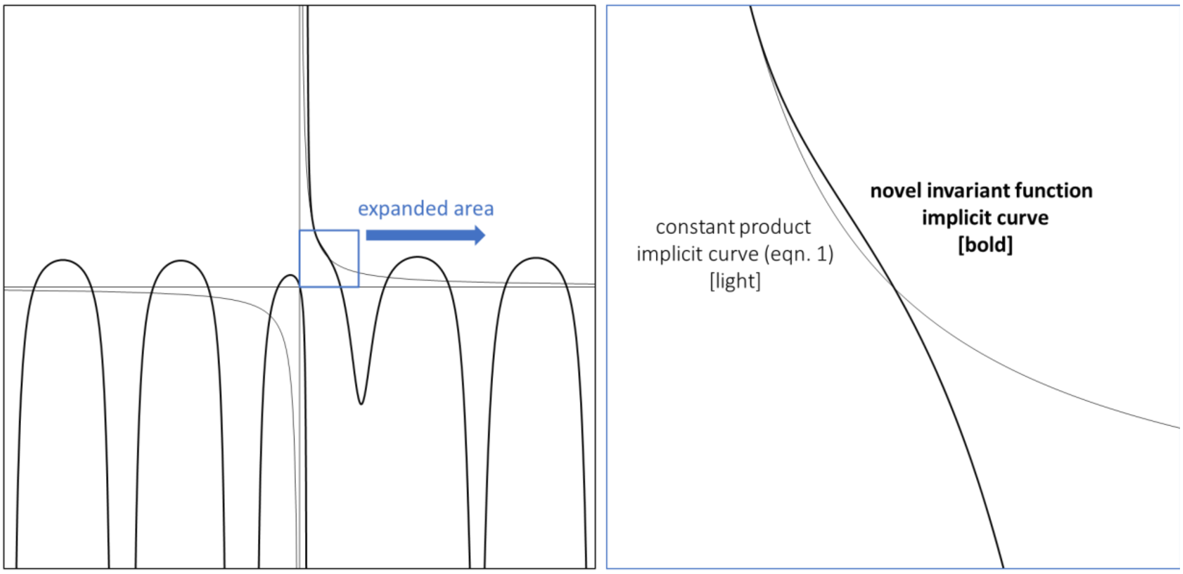



**Figure 8c.**

To further exemplify the scope of the invention, **eqns. 23a-c** substitutes *n* for $f(a, x, x_0, y, y_0)$. Consistent with its definition, and as has been observed for all examples examined herein, the novel function maintains a point on its graph with coordinates $x = x_0$, $y = y_0$ coincident with the graph of **eqn. 1**, and where the slope at this point is equal to *P* in both graphs (**Figure 9a-c**). Whereas the demonstration via **eqn. 22** is included for no other reason than to demonstrate the scope of the invariant function disclosed herein, **eqns. 23a-c** have special significance. It is a unique variation, and close homologue of the "stableswap" function currently in use by several DeFi protocols. The *a* term of **eqn. 23a** determines the degree to which the function is flattened around the point $x = x_0$, $y = y_0$. Higher values of *a* extend the flattened region, and at $a = 0$ the graph of the function is identical to that of **eqn. 1**.

$$n = \begin{cases} \dfrac{x_0^3}{x_0^3 - ax\left(x - 2x_0\right)} & x \leq x_0; y \geq y_0 \\[2ex] \dfrac{y_0^3}{y_0^3 - ay\left(y - 2y_0\right)} & x \geq x_0; y \leq y_0 \end{cases}$$ **eqn. 23a**

$$y = \begin{cases} \dfrac{k\left(x_0\left(2-n\right) - x\left(1-n\right)\right)}{x_0\left(nx + x_0\left(1-n\right)\right)} & x \leq x_0; y \geq y_0 \end{cases}$$ **eqn. 23b**

$$x = \begin{cases} \dfrac{x_0\left(k\left(2-n\right) - x_0 y\left(1-n\right)\right)}{k\left(1-n\right) + n x_0 y} & x \geq x_0; y \leq y_0 \end{cases}$$ **eqn. 23c**

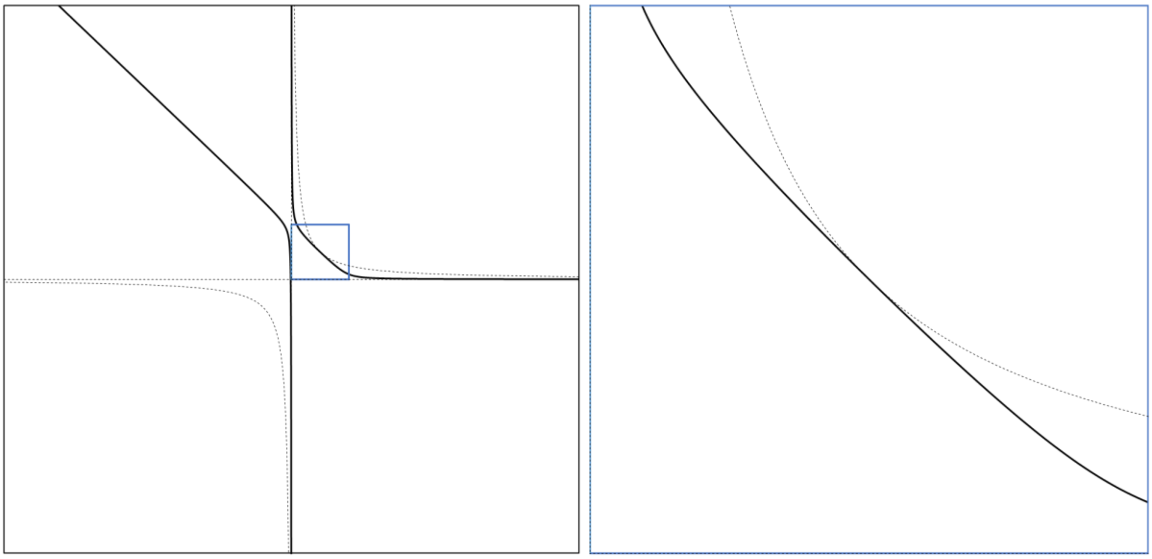


**Figure 9a.**

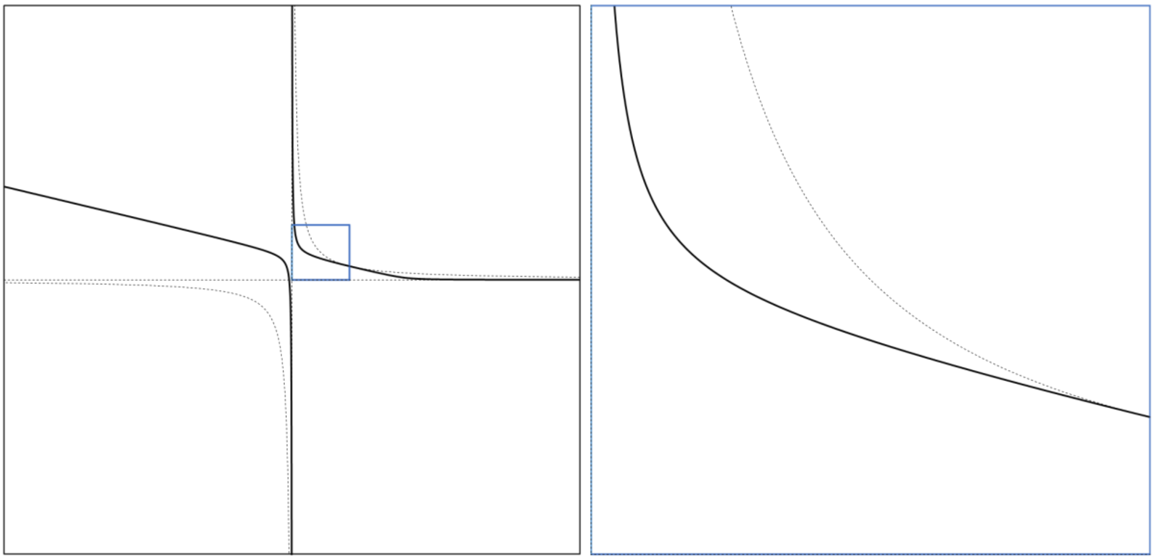


**Figure 9b.**

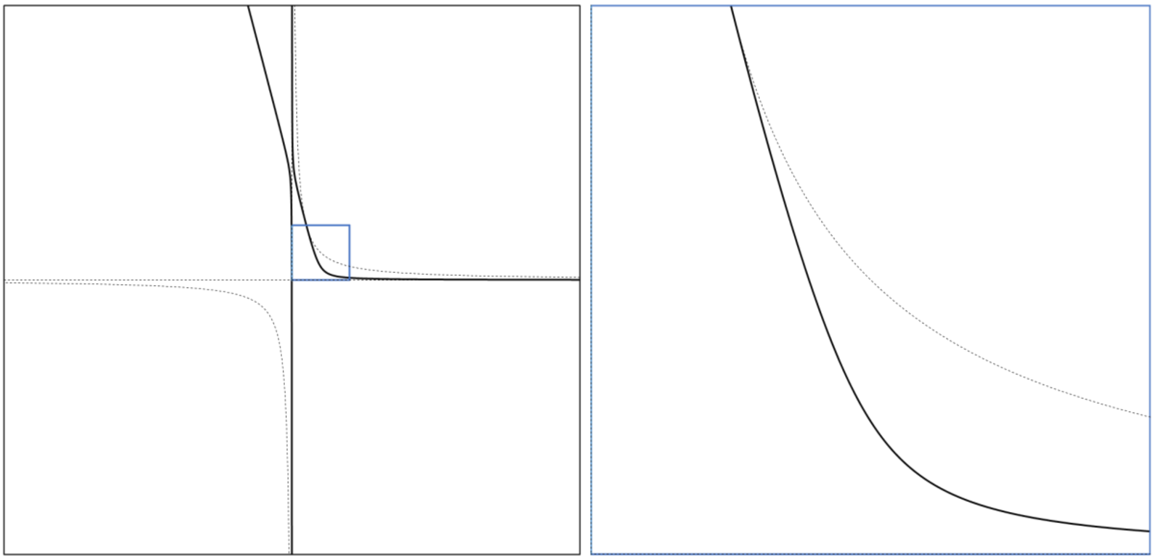


**Figure 9c.**

This concludes the section discussing the novel invariant function.

**Description of the Invention:** *Asymmetric Liquidity Pools*

The application of the novel invariant function in the context of AMMs and DEXes is now elaborated. The fundamental purpose of a DEX is to exchange tokens according to a preset pricing algorithm; the balances $x$ and $y$ are coordinates on the graph of the novel invariant function and represent all possible compositions of tokens and exchange rates. Therefore, any exchange can be represented as a secant line between any two coordinates in the positive domain, that describe the balance of $x$ and $y$ at two points in time, before and after the exchange occurred. Typically, the numeraire asset is assigned the $y$-axis, and the risk asset is assigned the $x$-axis, and this convention will be observed throughout the remainder of this document.

In one embodiment, the invariant function comprising a major component of the invention disclosed here can be used to create a price range through which one user can programmatically exchange their tokens for another of their choosing in a decentralized marketplace. In the common vernacular, an AMM position whereby tokens are exchanged only within predefined bounds of exchange rates is termed *concentrated liquidity*. In such a concentrated liquidity model, the graph of the invariant function in the positive domain necessarily intersects the $x$- and $y$-axis, and the slope of the graph at these intercepts defines the bounds of exchange rates. For any graph of the novel invariant function with asymptotes $x_{asym} < 0$ and $y_{asym} < 0$, the slope of the graph at $x = 0$, (i.e. the $y$-intercept, $y_{int}$) defines the highest exchange rate of the risk asset relative to its numeraire, and the slope of the graph at $y = 0$, (i.e. the $x$-intercept, $x_{int}$) defines the lowest exchange rate of the risk asset relative to its numeraire. For any graph of the novel invariant function where $n$, $x_{asym}$ and $y_{asym}$ are constant values, $x_{asym} < 0$ and $y_{asym} < 0$ if $n < 1$.

The following discussion expands on the application of the novel invariant function under exactly this pretense; however, this comprises only a subset of a wider scope of possible utilizations of the present invention, albeit an important one. Let the slope of the graph of the novel function at the $y$-intercept be $-P_a$ and the slope at the slope of the graph of the novel function at the $x$-intercept be $-P_b$. As with any permutation of the novel invariant function, the slope at the point coincident with **eqn. 1**, (i.e. $x = x_0$, $y = y_0$) is defined as $-P$. In one embodiment, any three of the parameters $P_a$, $P_b$, $y_{int}$ and $x_{int}$ can be user-provided, where $P_a$ and $P_b$ are the user's chosen range of exchange rates, $y_{int}$ represents the numeraire liquidity the user is committing to this same range, and $x_{int}$ represents the sum total of the risk asset the user has announced their intention to obtain via exchange with said numeraire.

The status quo in DeFi protocols employing concentrated liquidity models is that exchanges are reversible; users who have contributed tokens to the smart contract system exchange tokens in both directions concordant with the system state. This manifests as a symmetric exchange profile, save for a nominal fee, whereby users of these protocols are forced to accept the reverse exchange at the same rate as the forwards direction. In the present invention, the forwards and reverse exchange rates are separate, and entirely under the control of the user. Therefore, and in contrast to conventional AMMs, users' positions are described by two or more bonding curves, simultaneously. From this arrangement a novel capability of DeFi smart contract systems can emerge, which has no obvious parallel in competitor products. The asymmetric liquidity pool construction, in addition to a formal limit order functionality, offers what can best be described as support for automated trading strategies. Therefore, the term "user strategy" is used in the following sections to refer to a specific analogue of liquidity provision. It is helpful to interpret each strategy as its own unique liquidity pool vis-à-vis the entrenched meaning of the term. However, apart from both being smart contract systems that contain cryptocurrency tokens for the explicit purpose of performing exchange, the behavior of the strategies described herein are a profound shift from the standard DeFi archetypes.

The following illustrations refer to **eqns. 21a-i** from the previous section; however, what follows should be considered an example implementation. For completeness, eqns. **24a-l** demonstrate how any parameter discussed thus far can be computed from the example user inputs, $P_a$, $P_b$, and $y_{int}$, although only a subset, $B$, $S$, and $y_{int}$, are required for the remainder of this illustration.

$$x_{int} = \frac{y_{int}}{\sqrt{P_a P_b}} \qquad \textbf{eqn. 24a}$$

* $$B = \sqrt{P_b} \qquad \textbf{eqn. 24b}$$

$$P = \sqrt{P_a P_b} \qquad \textbf{eqn. 24c}$$

$$Q = \sqrt{\frac{P_b}{P_a}} \qquad \textbf{eqn. 24d}$$

$$R = \sqrt[4]{\frac{P_a}{P_b}} \qquad \textbf{eqn. 24e}$$

* $$S = \sqrt{P_a} - \sqrt{P_b} \qquad \textbf{eqn. 24f}$$

$$y_{asym} = \frac{\sqrt{P_b} y_{int}}{\sqrt{P_b} - \sqrt{P_a}} \qquad \textbf{eqn. 24g}$$

$$x_{asym} = \frac{\sqrt{P_b} y_{int}}{\sqrt{P_a P_b}\left(\sqrt{P_b} - \sqrt{P_a}\right)} \qquad \textbf{eqn. 24h}$$

$$k = \frac{y_{int}^2 \left(\sqrt[4]{P_a} - \sqrt[4]{P_b}\right)^2}{\sqrt{P_a}\left(\sqrt{P_a} - \sqrt{P_b}\right)^2} \qquad \textbf{eqn. 24i}$$

$$x_0 = \frac{y_{int}\left(\sqrt[4]{P_a P_b} - \sqrt{P_b}\right)}{\sqrt{P_a P_b}\left(\sqrt{P_a} - \sqrt{P_b}\right)} \qquad \textbf{eqn. 24j}$$

$$y_0 = \frac{y_{int}\left(\sqrt[4]{P_a P_b} - \sqrt{P_b}\right)}{\sqrt{P_a} - \sqrt{P_b}} \qquad \textbf{eqn. 24k}$$

$$n = 1 - \sqrt[4]{\frac{P_b}{P_a}} \qquad \textbf{eqn. 24l}$$

A user strategy is comprised of two or more sets of token balances, including token balances of zero, and two or more instances of the novel invariant function described in the previous section. For clarity of communication, the present discussion will be restricted to circumstances where a user has defined a strategy comprised of two tokens, and with two asymmetric bonding curves which determine their rate of exchange. However, scope of the invention has no such limitation.

To illustrate the emergent behavior of a user's strategy as a function of two asymmetric liquidity pools, cryptocurrency tokens designated as $USD_{TKN}$ and $RSK_{TKN}$ are used as a generic case. $USD_{TKN}$ is a stand-in for a "stablecoin", a cryptocurrency token that maintains an approximate 1:1 exchange value with the United States Dollar. The choice of a stable token was made for its intuitive properties which assist with the forthcoming explanations; however, the system is in no way dependent on such tokens to function. Similarly, the fictional characters Alice is used as a generic placeholder for any user with whom the protocol is interacting.

Assume that $RSK_{TKN}$ is currently trading at a value of \$1.00, and therefore a precise 1:1 exchange value with $TKN_{USD}$. Alice is a blockchain user who owns 100 $RSK_{TKN}$ and 5000 $USD_{TKN}$ and has ambitions to improve her overall quantity of both tokens over time by trading them against each other in an open marketplace. In its most elemental articulation, Alice is seeking to obtain additional $RSK_{TKN}$ when its exchange rate versus $USD_{TKN}$ is low, and then exchange it back for $USD_{TKN}$ when its exchange rate is high. Depending on her confidence, skills, knowledge, biases, predilections and other factors, Alice may have precise target rates of exchange with which she is comfortable replacing some quantity of $USD_{TKN}$ with $RSK_{TKN}$, and vice versa. In a conventional AMM, Alice has limited agency to exercise this most fundamental of economic activities. The issue arises from the fact that traditional AMM systems are symmetrical in their ability to facilitate exchange; if $USD_{TKN}$ is exchanged for some quantity of $RSK_{TKN}$, then the opposite exchange can also occur, regardless of Alice's pretension as a liquidity provider.

In the present invention, the exchange of $USD_{TKN}$ for $RSK_{TKN}$, and $RSK_{TKN}$ for $USD_{TKN}$, is prescribed by Alice, rather than the protocol. The exchange rate in each direction is determined by discrete instances of the novel invariant function, giving rise to a pair of bonding curves that allow for both tokens to be exchanged with participants in the marketplace at rates entirely independent of system state. In one aspect, Alice's liquidity position can represent a formal limit order, which in contrast to its counterpart in traditional finance, can be executed continuously over a premeditated range of exchange rates, as opposed to "all or nothing" given a single target rate at which the order executes. More importantly, the tokens with which Alice is executing her strategy are automatically available on its companion curve, at a different exchange rate to that with which they were acquired. The expression of Alice's objective occurs without her intervention or active management and represents a substantive improvement over existing DEX technologies.

The characteristic behavior of Alice's liquidity position (aka her strategy) in the context of the disclosed invention is now explored with reference to the novel invariant function presented in the previous section. First, consider the instance of the general formula where Alice is seeking to exchange her $RSK_{TKN}$ for $USD_{TKN}$ as its exchange rate moves towards Alice's targets. From the current market position of 1 $USD_{TKN}$ per $RSK_{TKN}$ (i.e. \$1.00), Alice wishes to begin exchanging her $RSK_{TKN}$ for $USD_{TKN}$ starting at a rate of 1½ $USD_{TKN}$ per $RSK_{TKN}$, up to a target of 3½ $USD_{TKN}$ per $RSK_{TKN}$, for which she is willing to contribute 100 $RSK_{TKN}$ to this range. By convention only, let the $y$-axis of Alice's bonding curve for this half of her strategy be represented by the quantity of $RSK_{TKN}$ she has remaining in said position. At the time she creates it, the first infinitesimal of $RSK_{TKN}$ is available to the market at an exchange rate of 1½ $USD_{TKN}$ per $RSK_{TKN}$. In one embodiment, the balance of $RSK_{TKN}$ at the time the position is created may define the $y_{int}$ parameter of the instance of the invariant function associated with Alice's strategy. Therefore, $P_a$, the gradient of the tangent line at $y_{int}$, defines the initial exchange rate. Since $P_a$ represents a change in $y$ relative to $x$, where $y$ denotes a true quantity of $RSK_{TKN}$, then $P_a$ is the reciprocal of Alice's chosen starting rate, assuming it was denominated in $USD_{TKN}$ as in the present example. In other words, the first infinitesimal change in $y$ with respect to $x$, concordant with Alice's mandate, is $-dy/dx$ = 1 $RSK_{TKN}$ / 1½ $USD_{TKN}$. Therefore $P_a = {}^2/_3$, as $USD_{TKN}$ is worth ${}^2/_3$ of $RSK_{TKN}$ at the first instance of exchange against Alice's liquidity. Thus, two of Alice's required inputs are $P_a = {}^2/_3$ and $y_{int} = 100$.

The final input for the first half of Alice's strategy, $P_b$, can be deduced via the same method as for $P_a$. The last infinitesimal amount of $RSK_{TKN}$, concordant with Alice's mandate, is to be exchanged at a rate of 3½ $USD_{TKN}$ per $RSK_{TKN}$, as her bonding curve for this range comes to rest at $x = x_{int}$, $y = 0$, thus fulfilling her order. Recall that $P_b$ is the gradient of the tangent line at this exact point on the curve, and represents a change in $y$ with respect to $x$. Therefore, its value is $-dy/dx$ = 1 $RSK_{TKN}$ / 3½ $USD_{TKN}$, or

$P_b = {}^2/_7$. With the third and final input in hand, Alice's bonding curve where $RSK_{TKN}$ is exchanged with the external market for $USD_{TKN}$ can be defined completely. From an implementation perspective, these three parameters are sufficient to define this half of the strategy; however, all previously discussed parameters have been calculated and are presented here for redundancy's sake (**Table 2**).

**Table 2.** The full parameter set describing one half of Alice's strategy, whereby her $RSK_{TKN}$ is exchanged with the external market for $USD_{TKN}$.

| **$RSK_{TKN}$ Curve Parameters and State** | |
|---|---|
| **Variable** | **Value** |
| $x$ | 0.0000000000000000 |
| $y$ | 100.0000000000000000 |
| $n$ | 0.1908932884297788 |
| $u$ | 5.2385288567536827 |
| $k$ | 4583.1375273971680144 |
| $B$ | 0.5345224838248488 |
| $P$ | 0.4364357804719848 |
| $Q$ | 0.6546536707079772 |
| $R$ | 1.2359309170224471 |
| $S$ | 0.2819740971028772 |
| $x_0$ | 102.4757889447313914 |
| $y_0$ | 44.7241009275762309 |
| $x_{asym}$ | -434.3465885608439976 |
| $y_{asym}$ | -189.5643923738959984 |
| $P_a$ | 0.6666666666666666 |
| $P_b$ | 0.2857142857142857 |
| $x_{int}$ | 229.1287847477919968 |
| $y_{int}$ | 100.0000000000000000 |

For the second half of Alice's strategy, assume she is motivated to exchange her $USD_{TKN}$ with the rest of the market for $RSK_{TKN}$ as its exchange rate approaches a value where she believes there is an opportunity.

From the current market position of 1 $USD_{TKN}$ per $RSK_{TKN}$ (i.e. $1.00), Alice wishes to begin exchanging her $USD_{TKN}$ for $RSK_{TKN}$ starting at a rate of ${}^2/_3$ $USD_{TKN}$ per $RSK_{TKN}$, down to a target of ${}^1/_5$ $USD_{TKN}$ per $RSK_{TKN}$, for which she is willing to contribute 5000 $USD_{TKN}$ to this range. The same convention applies to this bonding curve; let the $y$-axis of Alice's bonding curve for this half of her strategy be represented by the quantity of $USD_{TKN}$ she has remaining in said position. Note that both bonding curves have the true liquidity on the $y$-axis; the $x$-axis in both cases is just a coordinate, and not a token balance. At the time she creates her strategy, the first infinitesimal of $USD_{TKN}$ is available to the market at an exchange rate of 1 $USD_{TKN}$ per $RSK_{TKN}$. As before, assume the balance of $USD_{TKN}$ at the time the position is created defines the $y_{int}$ parameter for the instance of the invariant function associated with this half of Alice's strategy. The $P_a$ parameter of this half of the strategy, as was the case for the other curve, is the gradient of the tangent line at $y_{int}$, which defines the initial exchange rate in its cognate direction. Identical logic applies to the calculation of the curve parameters; however, note the flipped numeraire results in a more intuitive result in the case of $USD_{TKN}$ half of the strategy; there is no need to take a reciprocal in this case. Since $P_a$ represents a change in $y$ relative to $x$, where $y$ denotes a true quantity of $USD_{TKN}$, then $P_a$ is Alice's chosen rate as-is. The first infinitesimal change in $y$ with respect to $x$ is $-dy/dx = {}^2/_3$ $USD_{TKN}$ / 1 $RSK_{TKN}$. Therefore $P_a = {}^2/_3$, as $RSK_{TKN}$ is worth ${}^2/_3$ of $USD_{TKN}$ at the first instance of exchange against Alice's liquidity. Thus, two of Alice's required inputs are $P_a = {}^2/_3$ and $y_{int} = 5000$. As an aside, it is worth acknowledging that both curves now have identical $P_a$ values. However, their intrinsic meaning is reversed. In the first curve, $P_a$ denotes the initial rate at which $RSK_{TKN}$ is exchanged for $USD_{TKN}$; in the second curve, $P_a$ denotes the initial rate at which $USD_{TKN}$ is exchanged for $RSK_{TKN}$. In cash basis, $P_a$ on the first curve give rise to an initial valuation of $RSK_{TKN}$ \$1.50, whereas on the second curve the same $P_a$ value gives rise to an initial valuation of $RSK_{TKN}$ of \$0.66. In one embodiment, these and other impish nuances are hidden from the user through a front end interface, where inputs are collected in terms more aligned with their visceral understanding. The full description provided here should not be conflated with the user experience, but rather a detailed exploration of the application of the invariant function described in the previous section.

The final input for the second half of Alice's strategy, $P_b$, can be deduced via the same method as for $P_a$. The last infinitesimal amount of $USD_{TKN}$ is to be exchanged at a rate of ${}^1/_5$ $USD_{TKN}$ per $RSK_{TKN}$. Therefore, the $P_b$ value is $-dy/dx = {}^1/_5$ $USD_{TKN}$ / 1 $RSK_{TKN}$, and $P_b = {}^1/_5$. It is also worth observing that $P_a > P_b$ on both curves, which is expected, but not necessarily a requirement. This concludes the collection of the user inputs, and allows the second half of Alice's strategy to be defined, completing the creation of the strategy overall (**Table 3**).

**Table 3.** The full parameter set describing one half of Alice's strategy, whereby her $USD_{TKN}$ is exchanged with the external market for $RSK_{TKN}$.

| **$USD_{TKN}$ Curve Parameters and State** | |
|---|---|
| **Variable** | **Value** |
| $x$ | 0.0000000000000000 |
| $y$ | 5000.0000000000000000 |
| $n$ | 0.2599171955077147 |
| $u$ | 3.8473791549136593 |
| $k$ | 12384869.5652039740234613 |
| $B$ | 0.4472135954999579 |
| $P$ | 0.3651483716701107 |
| $Q$ | 0.5477225575051661 |
| $R$ | 1.3512001548070345 |
| $S$ | 0.3692829854277681 |
| $x_0$ | 5823.8614477945020553 |
| $y_0$ | 2126.5735244944962687 |
| $x_{asym}$ | -16582.7416875553499267 |
| $y_{asym}$ | -6055.1611250369005575 |
| $P_a$ | 0.6666666666666666 |
| $P_b$ | 0.2000000000000000 |
| $x_{int}$ | 13693.0639376291528606 |
| $y_{int}$ | 5000.0000000000000000 |

Depending on the environment, any subset or the complete set of values in **Tables 2** and **3** can be stored in a smart contract. However, in one embodiment, a frugal deployment will make use of only three parameters and a single $x$ or $y$ coordinate. For the example described here, and with reference to **eqns. 21a-i**, assume that only $y_{int}$, $B$, $S$, and the $y$-coordinate are used. Note that these choices reflect modest computational burden compared with the other parameters (**eqns. 24a-l**). Moreover, after the initial calculations of $B$ and $S$ during the creation of the strategy, which can be performed with the assistance of off-chain resources, all on-chain calculations performed during swaps use only the fundamental math operations addition, subtraction, multiplication, and division. This is true of all equations provided throughout this invention disclosure relating to the computation of marginal exchange rates and swaps, thus precluding the need for Taylor series or other polynomial approximations, or numeric methods of producing a function output. The freedom of choice to select appropriate parameters, and customized equation sets to fit the virtual machine on which the decentralized application is running is an important element, and further exemplifies the advantages of having the foundation of general theory as purported here.

Alice's strategy is now represented by two discrete bonding curves. In one embodiment, and in contrast to the conventional interpretation, only the $y$-axis of these curves has any intrinsic meaning; the $x$-axis does not denote a balance of tokens of any kind, virtual or otherwise. At the highest level of abstraction, the $x$-coordinate is a remnant of taking the antiderivative of Alice's desired exchange rate profile as a function of her true token balance, represented by $y$, implicit in the swap equations (**eqn. 21e**). The projection onto the $x$-axis gives rise to an additional dimension and creates the familiar looking bonding curve, but for the remaining demonstrations it should be stressed that the $x$-axis is effectively meaningless. Take special note that Alice has provided nonzero quantities of both $USD_{TKN}$ and $RSK_{TKN}$ tokens in the present example and has active nonzero balances in both tokens at this stage in the illustration, and yet the $x$-coordinate on both curves is nill. This is not a mistake. More importantly, the theory provided in the previous sections feature forms of critical equations including marginal exchange and effective exchange rates, and the swap equations themselves, which make no reference to the $x$-coordinate whatsoever. However, this is not a requirement, but only a nuance of the system as it is being described in the present context. In other embodiments, both axes may represent actual or virtual balances. Regardless, the familiarity of treating both axes as representing real token balances in a smart contract associated with a user's position has heuristic value, and despite its implicit irrelevance, still benefits the communication of more important concepts. As a compromise, the $x$-axis will be named only as $x$ in the forthcoming figures.

Before any exchanges are performed, Alice's pristine strategy is represented in **Figure 10**, where the top and bottom sets of curves are identical, but the axes have been rescaled in the latter from its square arrangement to allow for more prudent observation of changes to their structure during exchange. During the following discussion, changes in scale are necessary to make clear how each side of the strategy evolves under an asymmetric liquidity paradigm. At this stage in the discussion, the most important features of the curves are their intercepts and the $y$-coordinates. In the absence of any activity, both y-coordinates are at their starting points, their $y_{int}$ values, respectively. The $x$-intercepts are not incidental; these represent the number of tokens that will be acquired should the liquidity on the $y$-axis be completely depleted. The overall rate of exchange of the whole position – meaning the entire $y$-balance for its target, is equal to the geometric mean of the curve, which is the parameter $P$.

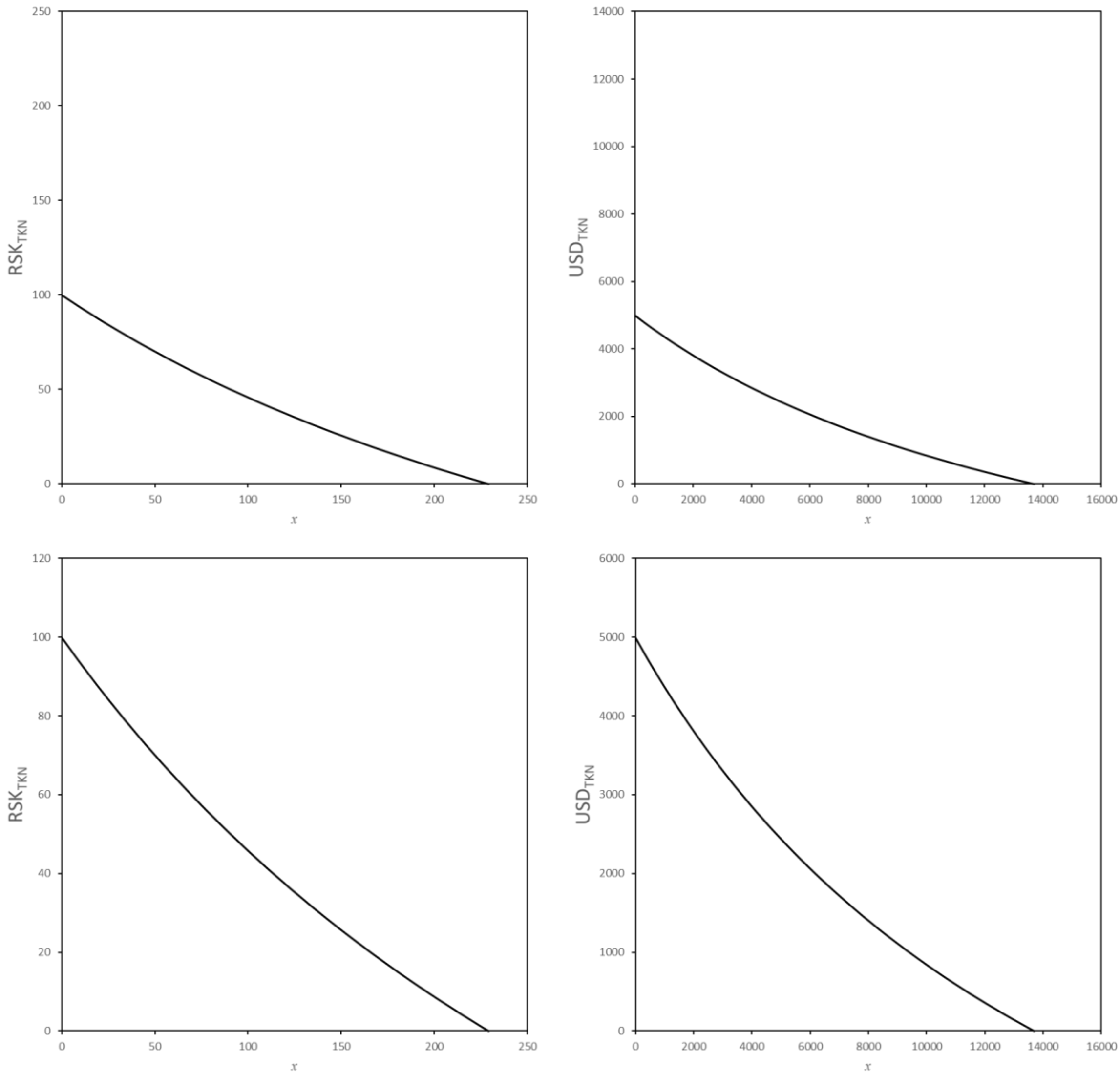



**Figure 10.**

Assume the market rate of RSK$_{TKN}$ declines with respect to USD$_{TKN}$ as Alice predicted, and market participants begin to interact with her strategy. Since Alice has an active strategy in USD$_{TKN}$, this is the liquidity source that supports market activity; her RSK$_{TKN}$ component is simply not attractive while quoting an exchange rate above that which the market is actively trading. To simplify this part of the discussion, the examples simulate a single bulk trade and sudden, violent changes in the accepted market rate; however, this obfuscates the time dimension, where many different actors may have interacted with Alice's strategy.

During a downturn in the market appraisal of RSK$_{TKN}$, assume its exchange rate versus USD$_{TKN}$ arrives at $^{12}/_{19}$, or \$0.63, which is inside of Alice's target. During this time, through arbitrage and other mechanisms, it can be assumed Alice's USD$_{TKN}$ liquidity will deplete to such a point where the exchange rate quoted by her bonding curve largely agrees with the rest of the market. To simulate this condition, the precise *y*-coordinate on the USD$_{TKN}$ curve can be computed from the marginal rate (**eqn 21e**). The *x*-coordinate can also be computed and is used in the following figures to indicate the overall progress towards fulfilling the order in its entirety. For the USD$_{TKN}$ curve, the only change apparent is the shift in the y-coordinate (**Table 4**).

**Table 4.** The full parameter set describing one half of Alice's strategy, whereby her $USD_{TKN}$ is exchanged with the external market for $RSK_{TKN}$. The numbers on the left side are those the position was commenced with at a time when the market appraisal of $USD_{TKN}$ to $RSK_{TKN}$ was at unity; the values on the right side represent the state of the strategy after a market downturn in the exchange rate of $RSK_{TKN}$ to $USD_{TKN}$ to $^{12}/_{19}$ ($0.63). The only changed value is that corresponding to the *y*-coordinate, and by extension, also the *x*-coordinate.

| **$USD_{TKN}$ Curve Parameters and State** | | |
|---|---|---|
| **Variable** | **Value** | |
| $x$ | 0.0000000000000000 | 454.4058235085825004 |
| $y$ | 5000.0000000000000000 | 4705.1425661613729972 |
| $n$ | 0.2599171955077147 | 0.2599171955077147 |
| $u$ | 3.8473791549136593 | 3.8473791549136593 |
| $k$ | 12384869.5652039740234613 | 12384869.5652039740234613 |
| $B$ | 0.4472135954999579 | 0.4472135954999579 |
| $P$ | 0.3651483716701107 | 0.3651483716701107 |
| $Q$ | 0.5477225575051661 | 0.5477225575051661 |
| $R$ | 1.3512001548070345 | 1.3512001548070345 |
| $S$ | 0.3692829854277681 | 0.3692829854277681 |
| $x_0$ | 5823.8614477945020553 | 5823.8614477945020553 |
| $y_0$ | 2126.5735244944962687 | 2126.5735244944962687 |
| $x_{asym}$ | -16582.7416875553499267 | -16582.7416875553499267 |
| $y_{asym}$ | -6055.1611250369005575 | -6055.1611250369005575 |
| $P_a$ | 0.6666666666666666 | 0.6666666666666666 |
| $P_b$ | 0.2000000000000000 | 0.2000000000000000 |
| $x_{int}$ | 13693.0639376291528606 | 13693.0639376291528606 |
| $y_{int}$ | 5000.0000000000000000 | 5000.0000000000000000 |

Inspection of the results above reveal an overall $\Delta x$ of 454.4058, representing a gain of $RSK_{TKN}$ to Alice's strategy, and $-\Delta y$ of 294.8574, corresponding to a loss of $USD_{TKN}$ from Alice's strategy, culminating in a cost base average of approximately $0.64 per $RSK_{TKN}$ acquired. Before examining the effect of the acquired $RSK_{TKN}$ on the state of Alice's strategy, first appreciate that this exchange is permanent. In standard AMMs, if market sentiment recovered after a sudden drawdown, Alice's liquidity position would rebalance, and thus the opportunity afforded to her by acquiring her target tokens at an attractive rate is rescinded (unless she is fast and attentive enough to respond appropriately). This is a significant change from the status quo. Alice's $USD_{TKN}$ curve *only* performs exchange in its forwards direction. Thus, the fate of the acquired $RSK_{TKN}$ provokes a justified curiosity.

The $RSK_{TKN}$ is absorbed by the strategy on its dedicated liquidity curve. More importantly, it is added to its *y*-axis. Since the balance of $RSK_{TKN}$ was already at parity with its *y*-intercept, the *y*-intercept is moved to accommodate the newfound liquidity. The parametrization of the novel invariant function comprising the first half of this invention disclosure is such that the $y_{int}$ value can be used as a type of container size; it can be moved without affecting the user's chosen exchange profile. In more explicit terms, the gradients at the *x*- and *y*-intercepts can remain constant, thus allowing the $RSK_{TKN}$ acquired at a low exchange rate to accumulate at the same price interval defined when the strategy was created. In the implementation considered for the present discussion, this effect can be achieved while only updating a single variable, the $y_{int}$ value, and therefore represents no significant computational overhead. Again, it must be stated that depending on the environment, if it is economical to maintain a dictionary of all parameters describing the pair of liquidity curves comprising Alice's strategy then so be it. The results of Alice's accumulation of $RSK_{TKN}$ in this example is summarized in **Table 5**. In contrast to the situation for the $USD_{TKN}$ curve, significant changes are apparent, particularly with respect to the natural scaling parameters, whereas the curvature parameters have remained unchanged.

**Table 5.** The full parameter set describing one half of Alice's strategy, whereby her $RSK_{TKN}$ is exchanged with the external market for $USD_{TKN}$. The numbers on the left side are those the position was commenced with at a time when the market appraisal of $USD_{TKN}$ to $RSK_{TKN}$ was at unity; the values on the right side represent the state of the strategy after a market downturn in the exchange rate of $RSK_{TKN}$ to USDTKN to $^{12}/_{19}$ ($0.63). The interval of exchange rates has not been affected, but only the relative size of this side of Alice's strategy.

| **$RSK_{TKN}$ Curve Parameters and State** | | |
|---|---|---|
| **Variable** | **Value** | |
| $x$ | 0.0000000000000000 | 0.0000000000000000 |
| $y$ | 100.0000000000000000 | 554.4058235085825572 |
| $n$ | 0.1908932884297788 | 0.1908932884297788 |
| $u$ | 5.2385288567536827 | 5.2385288567536827 |
| $k$ | 4583.1375273971680144 | 140869.9811174481583294 |
| $B$ | 0.5345224838248488 | 0.5345224838248488 |
| $P$ | 0.4364357804719848 | 0.4364357804719848 |
| $Q$ | 0.6546536707079772 | 0.6546536707079772 |
| $R$ | 1.2359309170224471 | 1.2359309170224471 |
| $S$ | 0.2819740971028772 | 0.2819740971028772 |
| $x_0$ | 102.4757889447313914 | 568.1317415959550772 |
| $y_0$ | 44.7241009275762309 | 247.9530200543385945 |
| $x_{asym}$ | -434.3465885608439976 | -2408.0427811921817920 |
| $y_{asym}$ | -189.5643923738959984 | -1050.9560306195387511 |
| $P_a$ | 0.6666666666666666 | 0.6666666666666666 |
| $P_b$ | 0.2857142857142857 | 0.2857142857142857 |
| $x_{int}$ | 229.1287847477919968 | 1270.3033259762037233 |
| $y_{int}$ | 100.0000000000000000 | 554.4058235085825572 |

The difference in the relative changes to either half of Alice's strategy is dramatic (**Figure 11**). The reason for this is that Alice's strategy is very top-heavy, culminating in what if often referred to as a "buy wall" on conventional order books. However, unlike a conventional orderbook, the tokens received through passive interaction with the outside market via one bonding curve are immediately and automatically added to its counterpart. Thus, a fully automated system for precise execution of premediated trading strategies is realized.

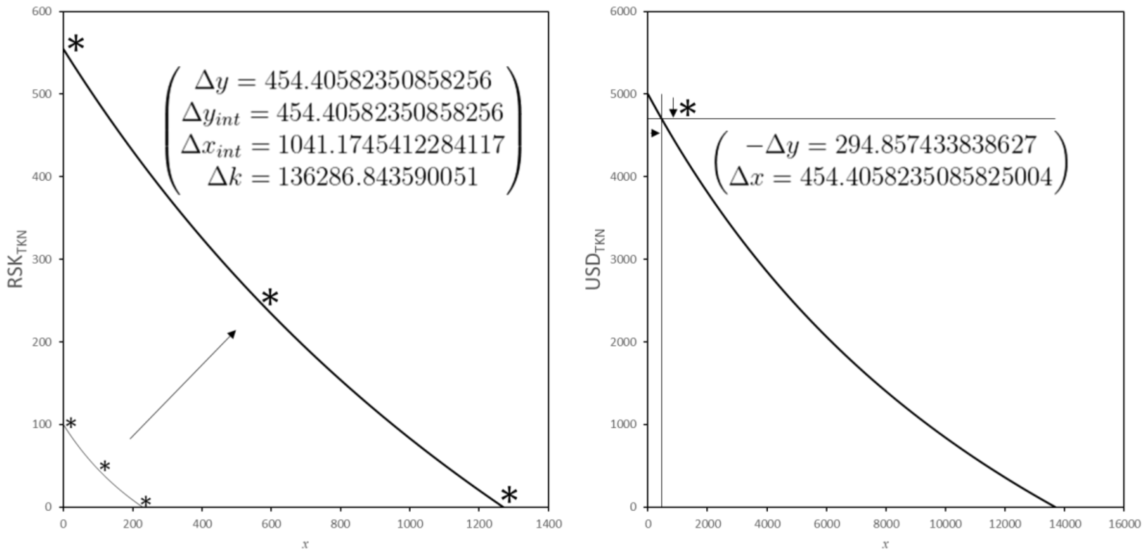



**Figure 11.**

As the exchange rate of $RSK_{TKN}$ continues to fall against $USD_{TKN}$, Alice's strategy will continue to part with its $USD_{TKN}$ in return for $RSK_{TKN}$. Assume the downturn continues to the point where the exchange rate of $RSK_{TKN}$ approaches $^9/_{25}$, or \$0.36, which remains inside of Alice's target range. Again, the effects of the changing market can be easily modelled by assuming the *y*-coordinate of the $USD_{TKN}$ side of her position comes to rest at the point where the gradient of its tangent line is precisely - $^9/_{25}$. This point can be computed from either the *x*- or *y*-coordinate via any of the marginal rate equations presented herein. As before, the $USD_{TKN}$ curve changes only with respect to the *y*-coordinate, which drags the *x*-coordinate behind it. The simulated drawdown is considerably more severe than what was considered in the previous example, consuming a more significant fraction of Alice's active limit order (**Table 6**). As an aside, the relative completion of Alice's limit order is not as trivial as it first appears. Measured from the number of $USD_{TKN}$ Alice has remaining, $1 - (y_{int} - y)/y_{int} = 41\%$; however, measured from the progression of the *x*-coordinate, $x/x_{int} = 44\%$, and $41\% \neq 44\%$.

It is important to note the tremendous change in scale resulting from the accumulation of additional $RSK_{TKN}$ by its cognate curve. This is trivial to achieve, as the parameter *k* has the same intrinsic meaning as in **eqns. 1a-c**. Comparing $\sqrt{k}$ for the $RSK_{TKN}$ curve from the instant of its creation up to the present moment reveals a 5.5×, followed by an 11.0× increase, culminating in an overall 60.8× increase in relative terms. Arbitrary changes in size are trivial, and do not give rise to performance or accuracy problems in the specific parametrization considered in this hypothetical. In part, this resiliency is owed to the fact that when a curve is expanding to accommodate additional liquidity, it can only do so such that $y = y_{int}$, precisely. Infinite relative growth in the value of *k* can also be achieved by allowing users to commence strategies containing only a single token balance, and therefore featuring an empty curve at its creation. This is not a ridiculous condition; strategies created with an empty curve (i.e. $y = y_{int} = 0$) must still contain information about the exchange interval, $\sqrt{P_a}$ and $\sqrt{P_b}$. Empty curves can begin to fill via the use of its companion's liquidity, in a manner identical to that demonstrated by Alice's $RSK_{TKN}$ curve. Moreover, users may wish to adjust the token balances on their strategies over time, including completely emptying one or both curves. None of these considerations present a problem for the embodiment of the invention presently under examination. The results of Alice's accumulation of $RSK_{TKN}$ in this example is summarized in **Table 7**, and the difference in the relative changes to either half of Alice's strategy is depicted in **Figure 12**.

**Table 6.** The full parameter set describing one half of Alice's strategy, whereby her $USD_{TKN}$ is exchanged with the external market for $RSK_{TKN}$. The values on the left side represent the state of the strategy after a market downturn in the exchange rate of $RSK_{TKN}$ to USDTKN to $^{12}/_{19}$ ($0.63); the values on the right side represent the state of the strategy after a market downturn in the exchange rate of $RSK_{TKN}$ to $USD_{TKN}$ to $^{9}/_{25}$ ($0.36).

| **$USD_{TKN}$ Curve Parameters and State** | | |
|---|---|---|
| **Variable** | **Value** | |
| $x$ | 454.4058235085825004 | 5983.5114629390054688 |
| $y$ | 4705.1425661613729972 | 2068.6900091410675486 |
| $n$ | 0.2599171955077147 | 0.2599171955077147 |
| $u$ | 3.8473791549136593 | 3.8473791549136593 |
| $k$ | 12384869.5652039740234613 | 12384869.5652039740234613 |
| $B$ | 0.4472135954999579 | 0.4472135954999579 |
| $P$ | 0.3651483716701107 | 0.3651483716701107 |
| $Q$ | 0.5477225575051661 | 0.5477225575051661 |
| $R$ | 1.3512001548070345 | 1.3512001548070345 |
| $S$ | 0.3692829854277681 | 0.3692829854277681 |
| $x_0$ | 5823.8614477945020553 | 5823.8614477945020553 |
| $y_0$ | 2126.5735244944962687 | 2126.5735244944962687 |
| $x_{asym}$ | -16582.7416875553499267 | -16582.7416875553499267 |
| $y_{asym}$ | -6055.1611250369005575 | -6055.1611250369005575 |
| $P_a$ | 0.6666666666666666 | 0.6666666666666666 |
| $P_b$ | 0.2000000000000000 | 0.2000000000000000 |
| $x_{int}$ | 13693.0639376291528606 | 13693.0639376291528606 |
| $y_{int}$ | 5000.0000000000000000 | 5000.0000000000000000 |

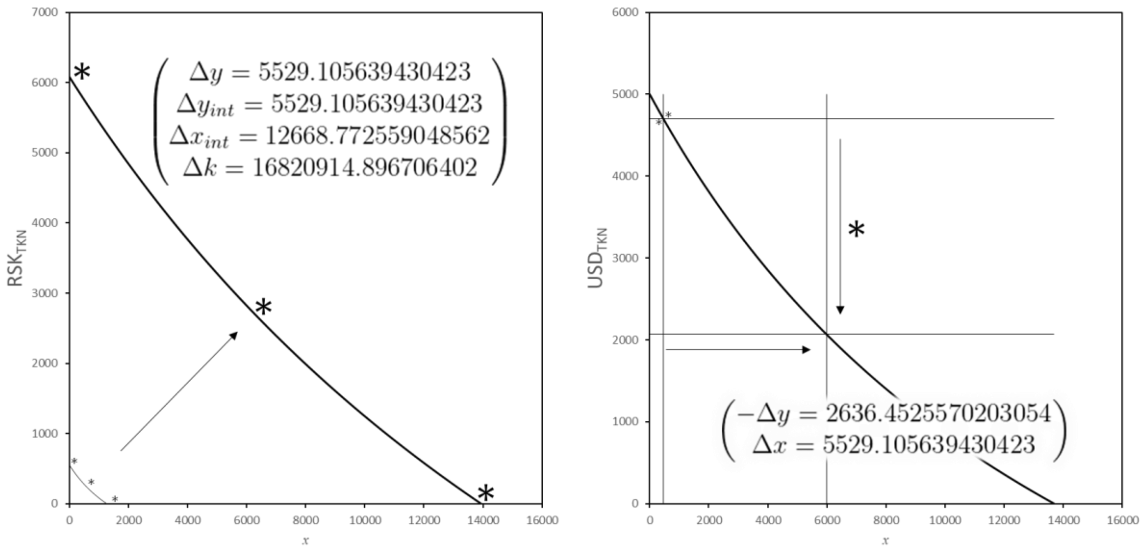



**Figure 12.**

**Table 7.** The full parameter set describing one half of Alice's strategy, whereby her $RSK_{TKN}$ is exchanged with the external market for $USD_{TKN}$. The values on the left side represent the state of the strategy after a market downturn in the exchange rate of $RSK_{TKN}$ to USDTKN to $^{12}/_{19}$ (\$0.63); the values on the right side represent the state of the strategy after a market downturn in the exchange rate of $RSK_{TKN}$ to $USD_{TKN}$ to $^{9}/_{25}$ (\$0.36).

| **$RSK_{TKN}$ Curve Parameters and State** | | |
|---|---|---|
| **Variable** | **Value** | |
| $x$ | 0.0000000000000000 | 0.0000000000000000 |
| $y$ | 554.4058235085825572 | 6083.5114629390054688 |
| $n$ | 0.1908932884297788 | 0.1908932884297788 |
| $u$ | 5.2385288567536827 | 5.2385288567536827 |
| $k$ | 140869.9811174481583294 | 16961784.8778238520026207 |
| $B$ | 0.5345224838248488 | 0.5345224838248488 |
| $P$ | 0.4364357804719848 | 0.4364357804719848 |
| $Q$ | 0.6546536707079772 | 0.6546536707079772 |
| $R$ | 1.2359309170224471 | 1.2359309170224471 |
| $S$ | 0.2819740971028772 | 0.2819740971028772 |
| $x_0$ | 568.1317415959550772 | 6234.1263671899168912 |
| $y_0$ | 247.9530200543385945 | 2720.7958066255100675 |
| $x_{asym}$ | -2408.0427811921817920 | -26423.5245039834626368 |
| $y_{asym}$ | -1050.9560306195387511 | -11532.1715397166371986 |
| $P_a$ | 0.6666666666666666 | 0.6666666666666666 |
| $P_b$ | 0.2857142857142857 | 0.2857142857142857 |
| $x_{int}$ | 1270.3033259762037233 | 13939.0758850247657392 |
| $y_{int}$ | 554.4058235085825572 | 6083.5114629390054688 |

Continuing with the examination of the asymmetric liquidity pools, assume that the local bottom in the exchange rate of $RSK_{TKN}$ to $USD_{TKN}$ was $^{9}/_{25}$ (\$0.36), after which its appraisal begins to improve. After an arbitrary period, a market rally carries the valuation of $RSK_{TKN}$ into Alice's target range. First, consider a moment in time where the exchange rate approached $1^{4}/_{5}$ $USD_{TKN}$ per $RSK_{TKN}$, or $^{5}/_{9}$ $RSK_{TKN}$ per $USD_{TKN}$ (i.e. \$1.80). The point on Alice's $RSK_{TKN}$ curve corresponding to $^{2}/_{5}$ can computed from the marginal rate (**eqn 21e**). For the first time in the present study, the $RSK_{TKN}$ curve has experienced at least one interaction with market participants, who have exchanged their $USD_{TKN}$ for Alice's $RSK_{TKN}$.

As anticipated, the *y*-coordinate (i.e. the true token balance) of Alice's $RSK_{TKN}$ curve has diminished (**Table 8**); while $USD_{TKN}$ has accumulated on its counterpart (**Table 9**). However, since Alice's $USD_{TKN}$ curve has ample space move its *y*-coordinate without exceeding $y_{int}$, no changes are made to its capacity parameters. Rather, the *y*-coordinate is simply moved to reflect the new $USD_{TKN}$ liquidity balance, which has a subsequent effect on the quoted instantaneous exchange rate for the first available infinitesimal within the curve. This is, of course, expected behavior. The exchange rate can only move within the bounds determined by Alice. In a manner of speaking, the accumulation of additional $USD_{TKN}$ at the current price point fills its cognate bonding curve from the bottom-up, slowly moving the reaccumulation target back towards the upper boundary (**Figure 13**).

**Table 8.** The full parameter set describing one half of Alice's strategy, whereby her $RSK_{TKN}$ is exchanged with the external market for $USD_{TKN}$. The values on the left side represent the state of the strategy after a market downturn in the exchange rate of $RSK_{TKN}$ to $USD_{TKN}$ to $^9/_{25}$ ($0.36); the values on the right side represent the state of the strategy after a market rally in the exchange rate of $RSK_{TKN}$ to $USD_{TKN}$ to $^5/_9$ ($1.80).

| **$RSK_{TKN}$ Curve Parameters and State** | | |
|---|---|---|
| **Variable** | **Value** | |
| $x$ | 0.0000000000000000 | 2521.9963352610334368 |
| $y$ | 6083.5114629390054688 | 4548.6733709747504690 |
| $n$ | 0.1908932884297788 | 0.1908932884297788 |
| $u$ | 5.2385288567536827 | 5.2385288567536827 |
| $k$ | 16961784.8778238520026207 | 16961784.8778238520026207 |
| $B$ | 0.5345224838248488 | 0.5345224838248488 |
| $P$ | 0.4364357804719848 | 0.4364357804719848 |
| $Q$ | 0.6546536707079772 | 0.6546536707079772 |
| $R$ | 1.2359309170224471 | 1.2359309170224471 |
| $S$ | 0.2819740971028772 | 0.2819740971028772 |
| $x_0$ | 6234.1263671899168912 | 6234.1263671899168912 |
| $y_0$ | 2720.7958066255100675 | 2720.7958066255100675 |
| $x_{asym}$ | -26423.5245039834626368 | -26423.5245039834626368 |
| $y_{asym}$ | -11532.1715397166371986 | -11532.1715397166371986 |
| $P_a$ | 0.6666666666666666 | 0.6666666666666666 |
| $P_b$ | 0.2857142857142857 | 0.2857142857142857 |
| $x_{int}$ | 13939.0758850247657392 | 13939.0758850247657392 |
| $y_{int}$ | 6083.5114629390054688 | 6083.5114629390054688 |

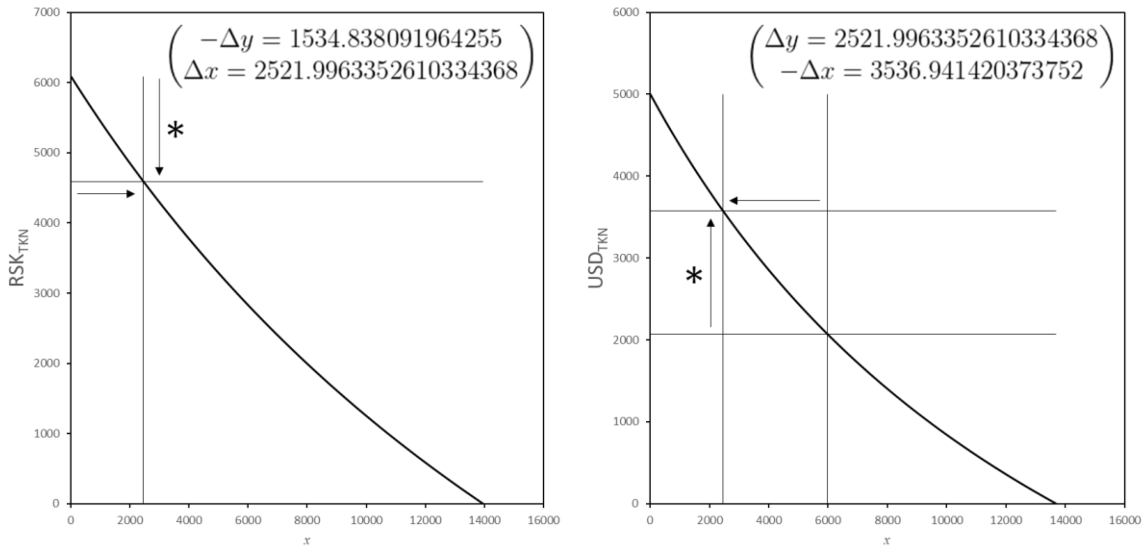



**Figure 13.**

**Table 9.** The full parameter set describing one half of Alice's strategy, whereby her $USD_{TKN}$ is exchanged with the external market for $RSK_{TKN}$. The values on the left side represent the state of the strategy after a market downturn in the exchange rate of $RSK_{TKN}$ to $USD_{TKN}$ to $^9/_{25}$ ($0.36); the values on the right side represent the state of the strategy after a market rally in the exchange rate of $RSK_{TKN}$ to $USD_{TKN}$ to $^5/_9$ ($1.80).

| **$USD_{TKN}$ Curve Parameters and State** | | |
|---|---|---|
| **Variable** | **Value** | |
| $x$ | 5983.5114629390054688 | 2446.5700425652537078 |
| $y$ | 2068.6900091410675486 | 4590.6863444021009854 |
| $n$ | 0.2599171955077147 | 0.2599171955077147 |
| $u$ | 3.8473791549136593 | 3.8473791549136593 |
| $k$ | 12384869.5652039740234613 | 12384869.5652039740234613 |
| $B$ | 0.4472135954999579 | 0.4472135954999579 |
| $P$ | 0.3651483716701107 | 0.3651483716701107 |
| $Q$ | 0.5477225575051661 | 0.5477225575051661 |
| $R$ | 1.3512001548070345 | 1.3512001548070345 |
| $S$ | 0.3692829854277681 | 0.3692829854277681 |
| $x_0$ | 5823.8614477945020553 | 5823.8614477945020553 |
| $y_0$ | 2126.5735244944962687 | 2126.5735244944962687 |
| $x_{asym}$ | -16582.7416875553499267 | -16582.7416875553499267 |
| $y_{asym}$ | -6055.1611250369005575 | -6055.1611250369005575 |
| $P_a$ | 0.6666666666666666 | 0.6666666666666666 |
| $P_b$ | 0.2000000000000000 | 0.2000000000000000 |
| $x_{int}$ | 13693.0639376291528606 | 13693.0639376291528606 |
| $y_{int}$ | 5000.0000000000000000 | 5000.0000000000000000 |

Finally, assume that at its local high, $RSK_{TKN}$ it achieved an exchange rate of 2½ $USD_{TKN}$ per $RSK_{TKN}$, or $^2/_5$ $RSK_{TKN}$ per $USD_{TKN}$ (i.e. $2.50), before capitulating back to a point outside of (i.e. between) both of Alice's designated price ranges, thus marking the completion of a market cycle from her perspective. For ease of analysis, let the final exchange rate between of $RSK_{TKN}$ and $USD_{TKN}$ be 1:1, thus having returned to the state whence this hypothetical began.

During the local top for $RSK_{TKN}$, the point on Alice's $RSK_{TKN}$ curve corresponding to $^2/_5$ can computed as demonstrated previously. As per the routine. the *y*-coordinate (i.e. the true token balance) of Alice's $RSK_{TKN}$ curve has further diminished due its valuation against $USD_{TKN}$ climbing farther into her range (**Table 8**); while $USD_{TKN}$ has accumulated on its counterpart curve (**Table 9**). Notably, this is the first case where Alice's balance of $USD_{TKN}$ exceeds the $y_{int}$ on its own curve; therefore, the capacity of the curve is increased to accommodate the new liquidity.

**Table 10.** The full parameter set describing one half of Alice's strategy, whereby her $RSK_{TKN}$ is exchanged with the external market for $USD_{TKN}$. The values on the left side represent the state of the strategy after a market rally in the exchange rate of $RSK_{TKN}$ to $USD_{TKN}$ to $^5/_9$ (\$1.80); the values on the right side represent the state of the strategy after a market rally in the exchange rate of $RSK_{TKN}$ to $USD_{TKN}$ to $^2/_5$ (\$2.50).

| **$RSK_{TKN}$ Curve Parameters and State** | | |
|---|---|---|
| **Variable** | **Value** | |
| $x$ | 2521.9963352610334368 | 7689.0989466937198813 |
| $y$ | 4548.6733709747504690 | 2112.8778405542366272 |
| $n$ | 0.1908932884297788 | 0.1908932884297788 |
| $u$ | 5.2385288567536827 | 5.2385288567536827 |
| $k$ | 16961784.8778238520026207 | 16961784.8778238520026207 |
| $B$ | 0.5345224838248488 | 0.5345224838248488 |
| $P$ | 0.4364357804719848 | 0.4364357804719848 |
| $Q$ | 0.6546536707079772 | 0.6546536707079772 |
| $R$ | 1.2359309170224471 | 1.2359309170224471 |
| $S$ | 0.2819740971028772 | 0.2819740971028772 |
| $x_0$ | 6234.1263671899168912 | 6234.1263671899168912 |
| $y_0$ | 2720.7958066255100675 | 2720.7958066255100675 |
| $x_{asym}$ | -26423.5245039834626368 | -26423.5245039834626368 |
| $y_{asym}$ | -11532.1715397166371986 | -11532.1715397166371986 |
| $P_a$ | 0.6666666666666666 | 0.6666666666666666 |
| $P_b$ | 0.2857142857142857 | 0.2857142857142857 |
| $x_{int}$ | 13939.0758850247657392 | 13939.0758850247657392 |
| $y_{int}$ | 6083.5114629390054688 | 6083.5114629390054688 |

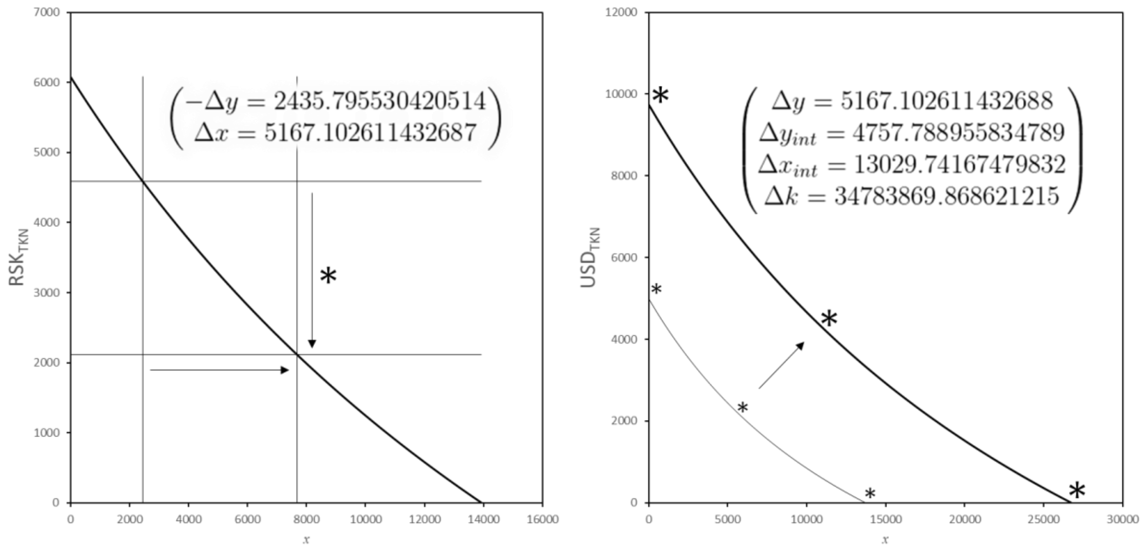



**Figure 14.**

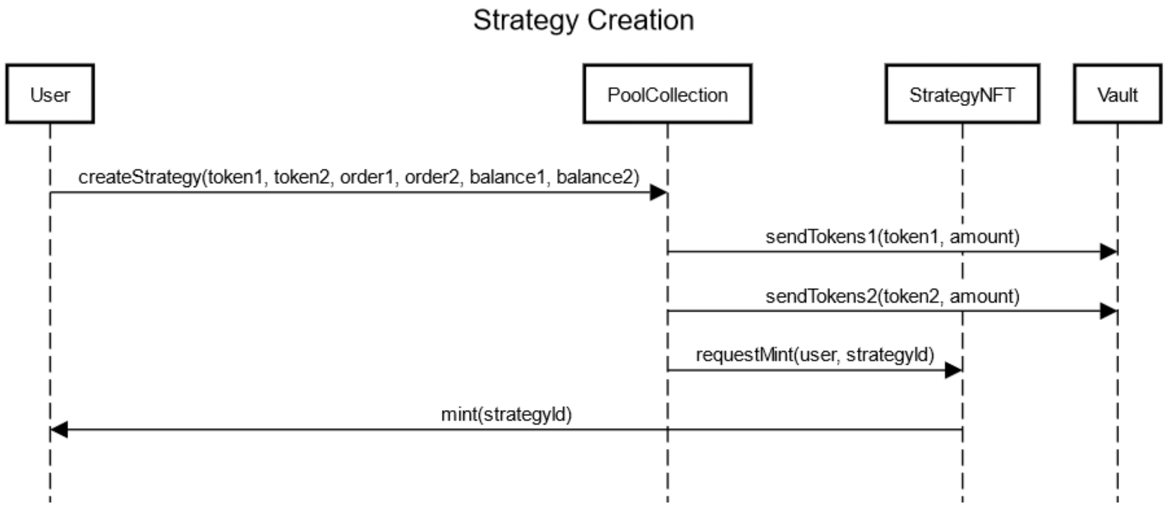
**Table 11.** The full parameter set describing one half of Alice's strategy, whereby her $USD_{TKN}$ is exchanged with the external market for $RSK_{TKN}$. The values on the left side represent the state of the strategy after a market rally in the exchange rate of $RSK_{TKN}$ to $USD_{TKN}$ to $^5/_9$ (\$1.80); the values on the right side represent the state of the strategy after a market rally in the exchange rate of $RSK_{TKN}$ to $USD_{TKN}$ to $^2/_5$ (\$2.50).

| **$USD_{TKN}$ Curve Parameters and State** | | |
|---|---|---|
| **Variable** | **Value** | |
| $x$ | 2446.5700425652537078 | 0.0000000000000000 |
| $y$ | 4590.6863444021009854 | 9757.7889558347887942 |
| $n$ | 0.2599171955077147 | 0.2599171955077147 |
| $u$ | 3.8473791549136593 | 3.8473791549136593 |
| $k$ | 12384869.5652039740234613 | 47168739.4338251873850822 |
| $B$ | 0.4472135954999579 | 0.4472135954999579 |
| $P$ | 0.3651483716701107 | 0.3651483716701107 |
| $Q$ | 0.5477225575051661 | 0.5477225575051661 |
| $R$ | 1.3512001548070345 | 1.3512001548070345 |
| $S$ | 0.3692829854277681 | 0.3692829854277681 |
| $x_0$ | 5823.8614477945020553 | 11365.6021831202397152 |
| $y_0$ | 2126.5735244944962687 | 4150.1311302166113819 |
| $x_{asym}$ | -16582.7416875553499267 | -32362.1787392577498395 |
| $y_{asym}$ | -6055.1611250369005575 | -11816.9968703370450385 |
| $P_a$ | 0.6666666666666666 | 0.6666666666666666 |
| $P_b$ | 0.2000000000000000 | 0.2000000000000000 |
| $x_{int}$ | 13693.0639376291528606 | 26722.8056124274735339 |
| $y_{int}$ | 5000.0000000000000000 | 9757.7889558347887942 |

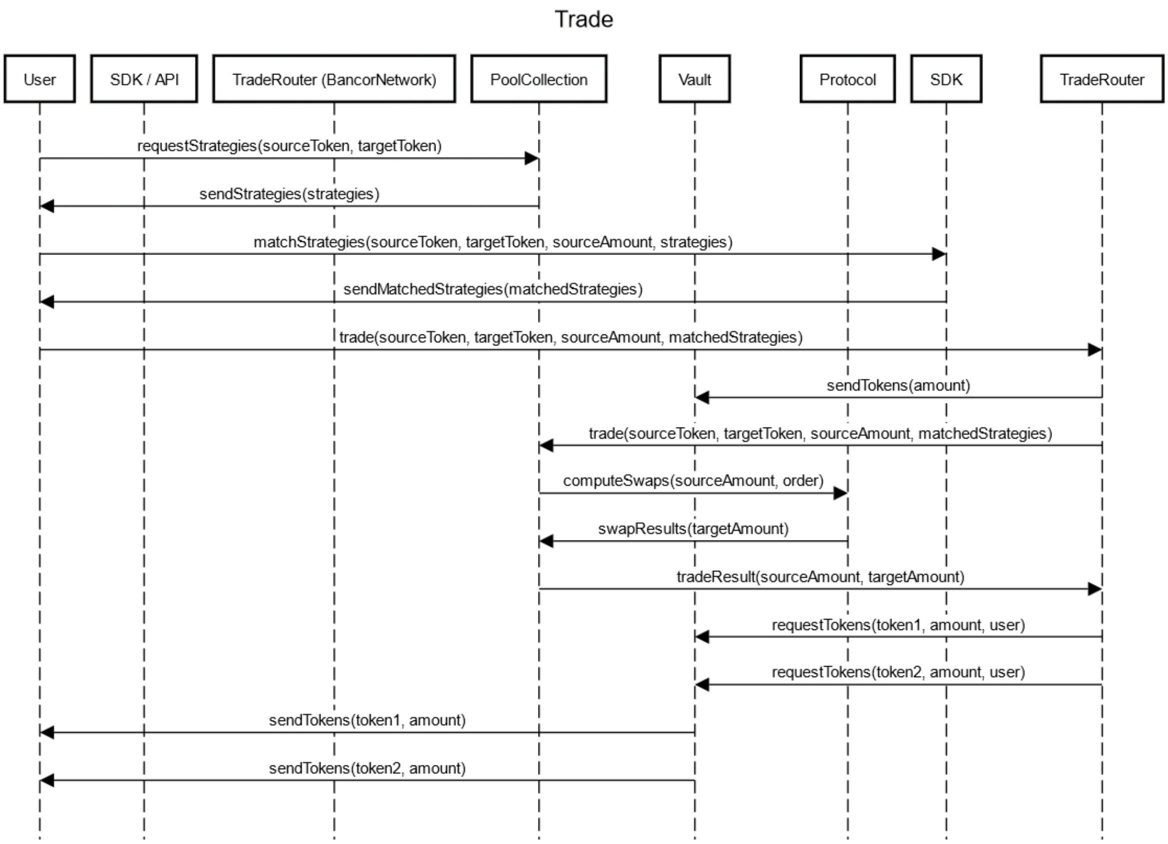
To conclude this demonstration, consider that Alice began with 100 $RSK_{TKN}$ and 5000 $USD_{TKN}$, at a price point where both were stated to be worth precisely \$1, totaling \$5,100 in value. Despite the absence of fee earnings of any kind, and without interacting with her position at all after instantiating it, Alice has improved her holdings of $RSK_{TKN}$ and $USD_{TKN}$ to 2112.88 and 9757.79, respectively, totaling \$11,870.67.

This concludes the section discussing asymmetric liquidity pools.

**Sequence Diagrams**




**Sequence Diagram 1.**




**Sequence Diagram 2.**

# Appendix

## 1. Derivation of the Constant Product swap formula from our prior art.

### First Swap

Equation for the following $F_t$tokens to be issued for the underlying $H_{t_1}$ tokens, with a $R_{r_1}$ reserve ratio and $T_{t_1}$ tokens in circulation (LP tokens in the pool) is **eqn 25**.

**eqn. 25**

$$F_t = \left| T_{t_1} \left( 1 + \frac{H_{t_1}}{T_{r_1}} \right)^{R_{r_1}} - T_{t_1} \right|$$

### Second Swap

Equation for how many $U_t$ tokens should be granted in exchange for the underlying $H_{t_2}$tokens, with a $R_{r_2}$ reserve ratio and $T_{t_2}$ tokens in circulation (LP tokens in the pool) is **eqn 26**.

**eqn. 26**

$$U_t = \left| T_{r_2} \sqrt[R_{r_2}]{1 + \frac{-H_{t_2}}{T_{t_2}}} - T_{r_2} \right|$$

### Deduction of the Full Swap Equation

A one-hop swap from any TKN1 to TKN2 is defined as taking two steps:

1. Issuing LP tokens that match the underlying TKN1 token to be swapped (source token).
2. The TKN2 tokens (target tokens) that should be granted in exchange for the underlying issued LP tokens.

Note that if the two steps occur in a single transaction, the LP token issuance happens atomically. For 50/50 pools, the Reserve Ratios $R_{r_1}$ and $R_{r_2}$ are equal to 50%.

The LP Token Supply $T_{t_1}$ and $T_{t_2}$ as well as the TKN1 and TKN2 Reserves $T_{r_1}$ and $T_{r_2}$ at any part of the swap, are always positive. The LP tokens issued amount $F_t$ is positive and the updated LP supply $T_{t_2}$ on step 2 consists of the sum between the previous LP Token Supply $T_{t_1}$ and the issued LP tokens $F_t$. Note that $T_{t_1}$ and $T_{t_2}$ are expressed in the same LP token numéraire.

The $H_{t_1}$ TKN1 Tokens sold for the LP token issuance is considered a positive number. That is, there will be an amount $T_{r_1}(k+1)$ at discrete time $(k+1)$ such that $T_{r_1}(k+1) = T_{r_1}(k) + H_{t_1}$. For the sake of redundancy and formula simplification, $T_{r_1}(k) = T_{r_1}$.

The $U_t$ TKN2 Tokens granted in exchange for the LP tokens issued is considered a positive number. That is, there will be an amount $T_{r_2}(k+1)$ at discrete time $(k+1)$ such that $T_{r_2}(k+1) = T_{r_2}(k) - U_t$. For the sake of redundancy and formula simplification, $T_{r_2}(k) = T_{r_2}$. These facts are expressed in the following equations and inequations for each step of the swap.

**First Step**

**eqn. 27**

$$\begin{cases} R_{r_1} = R_{r_2} = R_r = 50\% (Reserve\ Ratio\ Constant) \\ T_{t_1} > 0 (LP\ Token\ Supply) \\ T_{r_1} > 0 (TKN1\ Reserve) \\ F_t \geq 0 (LP\ Tokens\ to\ be\ issued) \\ H_{t_1} \geq 0 (TKNs\ to\ Swap) \end{cases}$$

The variables in the calculation of $F_t$ are replaced before simplification (**eqns 28a-c**).

**eqn. 28a**

$$F_t = \left| T_{t_1} \sqrt{1 + \frac{H_{t_1}}{T_{r_1}}} - T_{t_1} \right| = \left| T_{t_1} \left[ \sqrt{1 + \frac{H_{t_1}}{T_{r_1}}} - 1 \right] \right|$$

**eqn. 28b**

$$F_t = \left| T_{t_1} \left( 1 + \frac{H_{t_1}}{T_{r_1}} \right)^{R_{r_1}} - T_{t_1} \right| = \left| T_{t_1} \left( 1 + \frac{H_{t_1}}{T_{r_1}} \right)^{0.5} - T_{t_1} \right|$$

**eqn. 28c**

$$H_{t_1} \leq T_{r_1} \Rightarrow 1 + \frac{H_{t_1}}{T_{r_1}} \geq 1 \Rightarrow \sqrt{1 + \frac{H_{t_1}}{T_{r_1}}} - 1 \geq 0 \Rightarrow F_t = T_{t_1} \left[ \sqrt{1 + \frac{H_{t_1}}{T_{r_1}}} - 1 \right]$$

**Second Step**

**eqn. 29**

$$\begin{cases} T_{t_2} = T_{t_1} + F_t > 0 (Updated\ LP\ Token\ Supply) \\ T_{r_2} > 0 (TKN2\ Reserve) \\ H_{t_2} = F_t \geq 0 (LP\ Tokens\ issued) \\ U_t \geq 0 (TKNs\ Swapped) \end{cases}$$

The variables in the calculation of $U_t$ are replaced before simplification,

**eqn. 30a**

$$U_t = \left| T_{r_2} \sqrt[R_{r_2}]{1 + \frac{-H_{t_2}}{T_{t_2}}} - T_{r_2} \right| = \left| T_{r_2} \sqrt[0.5]{1 + \frac{-F_t}{T_{t_1} + F_t}} - T_{r_2} \right|$$

**eqn. 30b**

$$U_t = \left| T_{r_2} \left( \frac{T_{t_1}}{T_{t_1} + F_t} \right)^2 - T_{r_2} \right| = \left| T_{r_2} \left[ \left( \frac{T_{t_1}}{T_{t_1} + F_t} \right)^2 - 1 \right] \right|$$

**eqn. 31a**

$$\begin{cases} F_t \geq 0 \\ T_{t_1} > 0 \end{cases}$$

**eqn. 32b**

$$\Rightarrow 0 < \frac{T_{t_1}}{T_{t_1} + F_t} \leq 1$$

**eqn. 33c**

$$\Rightarrow \left( \frac{T_{t_1}}{T_{t_1} + F_t} \right)^2 - 1 \leq 0$$

**eqn. 34d**

$$\Rightarrow T_{r_2} \left[ \left( \frac{T_{t_1}}{T_{t_1} + F_t} \right)^2 - 1 \right] \leq 0$$

**eqn. 35e**

$$\Rightarrow U_t = T_{r_2} \left[ 1 - \left( \frac{T_{t_1}}{T_{t_1} + F_t} \right)^2 \right]$$

The final swap equation is **eqn 36**.

**eqn. 36**

$$U_t = T_{r_2} \frac{H_{t_1}}{T_{r_1} + H_{t_1}}$$

**Constant Product Swap Equation**

To demonstrate that a constant product curve using the $xy = k$ will result in the same swap formula for $U_t$, $x = x_1$ and $y = y_1$ express the TKN1 and TKN2 reserves before the swap, respectively; $x_2$ and $y_2$ represent the TKN1 and TKN2 reserves after the swap, respectively. To keep the swap equation with the same sign convention used previously, $H_{t_1} = \Delta x$ is a positive value and thus $x_2 = x_1 + \Delta x$ and $U_t = \Delta y$ is a positive value and thus $y_2 = y_1 - \Delta y$.

The aforementioned equations are repeated, and the simplified swap formula is obtained (**eqns 37a-d**).

**eqn. 37a**

$$\begin{cases} xy = k = x_1 y_1 = x_2 y_2 \\ y_2 = y_1 - \Delta y = y - \Delta y \\ x_2 = x_1 + \Delta x = x + \Delta x \end{cases}$$

**eqn. 37b**

$$\Rightarrow xy = (x + \Delta x)(y - \Delta y)$$

**eqn. 37c**

$$\Leftrightarrow xy = xy - x\Delta y + y\Delta x - \Delta x \Delta y$$

**eqn. 37d**

$$\Leftrightarrow \Delta y (x + \Delta x) = y\Delta x \Leftrightarrow \Delta y = \frac{y\Delta x}{(x + \Delta x)}$$

Replacing the used naming convention in the swap equation for $\Delta y$ with **eqn 38**, one obtains the previously calculated formula for $U_t$ (**eqn 39**).

**eqn. 38**

$$\begin{cases} U_t = \Delta y \\ T_{r_1} = x \\ T_{r_2} = y \\ H_{t_1} = \Delta x \end{cases}$$

**eqn. 39**

$$\Delta y = \frac{y\Delta x}{(x + \Delta x)} \Leftrightarrow U_t = T_{r_2} \frac{H_{t_1}}{T_{r_1} + H_{t_1}} \blacksquare$$